ON THE ABSOLUTE CESÀRO SUMMABILITY OF INTEGRALS

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PART 1

1. WE write,* for $t \geqslant 1$,

$$\left.\begin{aligned} I_\alpha f(t) = f_\alpha(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t (t-u)^{\alpha-1} f(u)\, du \quad (\alpha > 0), \\ I_0 f(t) = f_0(t) &= f(t); \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} \phi^{(\delta)}(t,x) &= \frac{1}{\Gamma(1-\delta)} \frac{d}{dt} \int_t^x (u-t)^{-\delta} \phi(u)\, du \quad (0 < \delta < 1,\, x > t), \\ D^\delta \phi(t) = \phi^{(\delta)}(t) &= \lim_{x \to \infty} \phi^{(\delta)}(t,x) \quad (0 < \delta < 1), \\ D^0 \phi(t) = \phi^{(0)}(t) &= \phi(t), \\ D^{\delta+s}\phi(t) = \phi^{(\delta+s)}(t) &= (d/dt)^s \phi^{(\delta)}(t) \\ &\qquad (0 \leqslant \delta < 1,\ s \text{ a positive integer}). \end{aligned}\right\} \tag{1.2}$$

2. The object of Part 1 of this paper is to prove the following theorem.

THEOREM 1. (*a*) *If* $\lambda \geqslant 0$, *and if*

(i) $\int_1^\infty f(t)\, dt$ *is summable* $|C, \lambda|$,†

(ii) $\phi(t)$ *is essentially bounded in* $(1, \infty)$,

(iii) $\phi^{(\lambda)}(t)$ *is absolutely continuous*,‡

(iv) $t^\lambda \phi^{(\lambda)}(t) = O(1)$ *in* $(1, \infty)$,

then $\int_1^\infty f(t)\phi(t)\, dt$ *is summable* $|C, \lambda|$.

(*b*) *The conclusion remains valid if* λ *is replaced by any integer greater than* λ *in hypotheses* (iii) *and* (iv).

* Throughout this paper $f(t)$ and $\phi(t)$ denote functions integrable in the Lebesgue sense in every finite interval in $(1, \infty)$. Every integral over a finite range is a Lebesgue integral, and $\int^\infty$ denotes $\lim_{x\to\infty} \int^x$, if this limit exists, finite or infinite.

† i.e. $t^{-\lambda} f_{\lambda+1}(t)$ is of bounded variation in $(1, \infty)$.

‡ Where no interval of absolute continuity is specified it is to be understood that the property pertains to every finite interval in $(1, \infty)$.

Corresponding theorems for summability (C, λ) have been given by Hardy* and Cossar,† who dealt respectively with integral and non-integral λ. Fekete‡ and Bosanquet§ have established analogous results for series summable $|C, \lambda|$, where λ is an integer. Conditions (ii) and (iv) were suggested by similar conditions appearing in their papers. With reference to version (*a*) of the theorem, it will be proved in Part 2 that these two conditions are the best possible when condition (iii) is satisfied. We shall require the following lemmas.

3. LEMMA 1. *If* $0 < \delta < 1$, $\phi(t)$ *is essentially bounded in* $(1, \infty)$ *and* $\phi^{(\delta)}(t)$ *is absolutely continuous, then there is an absolutely continuous function* $\psi(t)$ *such that*

(i) $\psi(t) = \phi(t)$ *p.p. in* $(1, \infty)$;

(ii) $$\psi(t) = -\frac{1}{\Gamma(\delta)} \int_t^x (u-t)^{\delta-1} \phi^{(\delta)}(u)\, du + \frac{\delta}{\Gamma(\delta)\Gamma(1-\delta)} \int_t^x (u-t)^{\delta-1}\, du \int_x^\infty (v-u)^{-\delta-1} \phi(v)\, dv,$$

for $t \geqslant 1$ *and any* $x > t$.

Write, for $x > t \geqslant 1$,

$$\psi(t) = -\frac{1}{\Gamma(\delta)} \int_t^x (u-t)^{\delta-1} \phi^{(\delta)}(u,x)\, du. \tag{3.1}$$

It has been proved elsewhere|| that $\psi(t)$ is independent of x, and that it is an absolutely continuous function equivalent to $\phi(t)$.

Further, since $\phi(t)$ is essentially bounded in $(1, \infty)$, it follows from the definitions (1.2) that, for $x > u \geqslant 1$,

$$\phi^{(\delta)}(u) = \phi^{(\delta)}(u,x) + \frac{\delta}{\Gamma(1-\delta)} \int_x^\infty (v-u)^{-\delta-1} \phi(v)\, dv. \tag{3.2}$$

Result (ii) is now obtained from (3.1) and (3.2).

LEMMA 2. *If* $\lambda > 1$, $\phi(t)$ *is essentially bounded in* $(1, \infty)$ *and*

$$t^\lambda \phi^{(\lambda)}(t) = O(1)$$

in $(1, \infty)$, *then, for* $r = 1, 2, \ldots, [\lambda]$,

$$t^{\lambda-r} \phi^{(\lambda-r)}(t) = O(1) \text{ in } (1, \infty).\dagger\dagger$$

* G. H. Hardy, *Messenger of Math.* 40 (1911), 87–91 and 108–12.

† J. Cossar, *Journal London Math. Soc.* 16 (1941), 56–68.

‡ M. Fekete, *Math. és Termés Ért.* 35 (1917), 309–24.

§ L. S. Bosanquet (**4**).

|| D. Borwein (**2**), Lemma 7.

†† Cf. Bosanquet (**4**), Lemma 8.

Let $s = [\lambda]$, $\delta = \lambda - s$. Clearly $\int_1^\infty \phi^{(\lambda)}(t)\,dt$ converges, and thus, as $t \to \infty$, $\phi^{(\lambda-1)}(t) \to l$, a finite number. Hence, in view of a well-known property of Cesàro means, we have

$$\frac{l}{\Gamma(s)} = \lim_{t\to\infty} t^{1-s} I_{s-1}\,\phi^{(\lambda-1)}(t) = \lim_{t\to\infty} t^{1-s}\phi^{(\delta)}(t). \tag{3.3}$$

When λ is an integer ($s \geqslant 2$, $\delta = 0$), $\phi(t) = \phi^{(0)}(t)$ is continuous for $t \geqslant 1$, and hence it follows from (3.3) and the first hypothesis on $\phi(t)$ that $l = 0$.

When λ is not an integer ($0 < \delta < 1$), we deduce from Lemma 1 (ii) that

$$\int_1^x (u-1)^{\delta-1}\phi^{(\delta)}(u)\,du = O(1) \text{ in } (1,\infty). \tag{3.4}$$

Since $s \geqslant 1$, (3.3) is compatible with (3.4) only if $l = 0$. Hence

$$\phi^{(\lambda-1)}(t) = -\int_t^\infty \phi^{(\lambda)}(u)\,du = O(t^{1-\lambda}) \text{ in } (1,\infty).$$

Repetition of the above argument, if necessary, yields the result.

LEMMA 3. *If* $0 < \delta < 1$, $\phi(t)$ *is absolutely continuous and is bounded in* $(1,\infty)$, *and* $t^{\delta+1}\phi^{(\delta+1)}(t) = O(1)$ *in* $(1,\infty)$, *then, for* $t \geqslant 1$,

(i) $\displaystyle \frac{1}{\Gamma(\delta)}\int_t^\infty (u-t)^{\delta-1}\phi^{(\delta+1)}(u)\,du = -\phi'(t)$,

(ii) $t\phi'(t) = O(1)$,

(iii) $\displaystyle \frac{1}{\Gamma(1-\delta)}\int_t^\infty (u-t)^{-\delta}\phi'(u)\,du = \phi^{(\delta)}(t)$,

(iv) $D^\delta\{t\phi'(t)\} = \delta\phi^{(\delta)}(t) + t\phi^{(\delta+1)}(t)$.

Suppose that $1 \leqslant t < x$. Write, for $1 \leqslant u < 2x$,

$$\theta(u) = \frac{\phi^{(\delta)}(u)}{\Gamma(\delta)} - \frac{\delta}{\Gamma(\delta)\Gamma(1-\delta)}\int_{2x}^\infty (v-u)^{-\delta-1}\phi(v)\,dv. \tag{3.5}$$

Since $\phi^{(\delta+1)}(u)$ exists and $\phi(u)$ is bounded, for $1 \leqslant u < \infty$, we have, for $1 \leqslant u < 2x$,

$$\theta'(u) = \frac{\phi^{(\delta+1)}(u)}{\Gamma(\delta)} - \frac{\delta(\delta+1)}{\Gamma(\delta)\Gamma(1-\delta)}\int_{2x}^\infty (v-u)^{-\delta-2}\phi(v)\,dv. \tag{3.6}$$

Clearly $\theta'(u)$ exists and is bounded in $(1,x)$.

Let
$$M = \operatorname*{bound}_{u\geqslant 1}\left\{\frac{u^\delta|\phi^{(\delta)}(u)|}{\Gamma(\delta)} + \frac{|\phi(u)|}{\Gamma(\delta)\Gamma(1-\delta)}\right\}. \tag{3.7}$$

In view of Lemma 2, M is finite.

It follows from (3.5) and (3.7) that, for $1 \leqslant u < 2x$,

$$|\theta(u)| \leqslant M\{u^{-\delta} + (2x-u)^{-\delta}\}. \tag{3.8}$$

In virtue now of Lemma 1 (ii) and (3.5), we have

$$\phi(t) = -\int_t^{2x} (u-t)^{\delta-1}\theta(u)\,du.$$

Hence

$$\phi(t) + \int_x^{2x} (u-t)^{\delta-1}\theta(u)\,du + (x-t)^\delta\theta(x)/\delta$$
$$= \int_t^x (u-t)^{\delta-1}\,du \int_u^x \theta'(w)\,dw = \int_t^x du \int_u^x (w-u)^{\delta-1}\theta'(w)\,dw. \tag{3.9}$$

Since $\theta'(w)$ is bounded in $(1,x)$, and, by (3.8), $\theta(u)$ is integrable in $(x,2x)$, we obtain, on differentiating (3.9) with respect to t and then applying (3.6),

$$\phi'(t) + (1-\delta)\int_x^{2x} (u-t)^{\delta-2}\theta(u)\,du - (x-t)^{\delta-1}\theta(x)$$
$$= -\int_t^x (w-t)^{\delta-1}\theta'(w)\,dw$$
$$= -\frac{1}{\Gamma(\delta)}\int_t^x (w-t)^{\delta-1}\phi^{(\delta+1)}(w)\,dw +$$
$$+ \frac{\delta(\delta+1)}{\Gamma(\delta)\Gamma(1-\delta)}\int_t^x (w-t)^{\delta-1}\,dw \int_{2x}^\infty (v-w)^{-\delta-2}\phi(v)\,dv. \tag{3.10}$$

Further, by (3.7) and (3.8),

$$(1-\delta)\int_x^{2x} (u-t)^{\delta-2}|\theta(u)|\,du + (x-t)^{\delta-1}|\theta(x)| +$$
$$+ \frac{\delta(\delta+1)}{\Gamma(\delta)\Gamma(1-\delta)}\int_t^x (w-t)^{\delta-1}\,dw \int_{2x}^\infty (v-w)^{-\delta-2}|\phi(v)|\,dv$$
$$\leqslant 2M(1-\delta)(x-t)^{-2}\int_x^{2x} (2x-u)^{-\delta}\,du + 2M(x-t)^{\delta-1}x^{-\delta} +$$
$$+ M\delta(\delta+1)\int_t^x (w-t)^{\delta-1}\,dw \int_{2x}^\infty (v-x)^{-\delta-2}\,dv$$
$$= 2M(x-t)^{\delta-2}x^{1-\delta} + 2M(x-t)^{\delta-1}x^{-\delta} + M(x-t)^\delta x^{-\delta-1}$$
$$= o(1) \text{ as } x \to \infty. \tag{3.11}$$

Conclusion (i) follows from (3.10) and (3.11); and conclusion (ii) is an immediate consequence of (i) and the final hypothesis. We can also deduce (iii) from (i) as follows:

$$\frac{1}{\Gamma(1-\delta)}\int\limits_t^\infty (u-t)^{-\delta}\phi'(u)\,du$$

$$= \frac{-1}{\Gamma(1-\delta)\Gamma(\delta)}\int\limits_t^\infty (u-t)^{-\delta}\,du \int\limits_u^\infty (v-u)^{\delta-1}\phi^{(\delta+1)}(v)\,dv$$

$$= \frac{-1}{\Gamma(1-\delta)\Gamma(\delta)}\int\limits_t^\infty \phi^{(\delta+1)}(v)\,dv\int\limits_t^v (u-t)^{-\delta}(v-u)^{\delta-1}\,du$$

$$= -\int\limits_t^\infty \phi^{(\delta+1)}(v)\,dv = \phi^{(\delta)}(t);$$

the inversion of the order of integration and the final equality being justified since $\phi^{(\delta+1)}(v) = O(v^{-\delta-1})$ in $(1,\infty)$ and, by Lemma 2, $\phi^{(\delta)}(v) = o(1)$ as $v \to \infty$.

To complete the proof of the lemma we observe that

$$\int\limits_t^x (u-t)^{-\delta}u\phi'(u)\,du - t\int\limits_t^x (u-t)^{-\delta}\phi'(u)\,du$$

$$= \int\limits_t^x (u-t)^{1-\delta}\phi'(u)\,du = (1-\delta)\int\limits_t^x (u-t)^{-\delta}\,du\int\limits_u^x \phi'(v)\,dv$$

$$= (\delta-1)\int\limits_t^x (u-t)^{-\delta}\phi(u)\,du + (x-t)^{1-\delta}\phi(x).$$

Hence, by result (iii),

$$\frac{d}{dt}\int\limits_t^x (u-t)^{-\delta}u\phi'(u)\,du$$

$$= \frac{d}{dt}\Big\{\Gamma(1-\delta)t\phi^{(\delta)}(t) - t\int\limits_x^\infty (u-t)^{-\delta}\phi'(u)\,du\Big\} +$$

$$+(\delta-1)\frac{d}{dt}\int\limits_t^x (u-t)^{-\delta}\phi(u)\,du + (\delta-1)(x-t)^{-\delta}\phi(x)$$

$$= \Gamma(1-\delta)\{t\phi^{(\delta+1)}(t)+\phi^{(\delta)}(t)\} + (\delta-1)\frac{d}{dt}\int\limits_t^x (u-t)^{-\delta}\phi(u)\,du +$$

$$+(\delta-1)(x-t)^{-\delta}\phi(x) - \frac{d}{dt}\Big\{t\int\limits_x^\infty (u-t)^{-\delta}\phi'(u)\,du\Big\}. \tag{3.12}$$

Since $\phi(x) = O(1)$ in $(1,\infty)$, $\phi(x)(x-t)^{-\delta} = o(1)$ as $x\to\infty$; and, since $u\phi'(u) = O(1)$ in $(1,\infty)$,

$$\frac{d}{dt}\Big\{t\int\limits_x^\infty (u-t)^{-\delta}\phi'(u)\,du\Big\} = \int\limits_x^\infty (u-t)^{-\delta}\phi'(u)\,du + \delta t\int\limits_x^\infty (u-t)^{-\delta-1}\phi'(u)\,du$$

$$= O\Big\{\int\limits_x^\infty (u-x)^{-\delta}u^{-1}\,du + tx^{-1}\int\limits_x^\infty (u-t)^{-\delta-1}\,du\Big\}$$

$$= O\{x^{-\delta}+tx^{-1}(x-t)^{-\delta}\}$$

$$= o(1) \text{ as } x\to\infty.$$

Consequently conclusion (iv) follows from (3.12) and the definitions (1.2).

LEMMA 4. *If s is a positive integer, $\phi^{(s)}(t)$ is absolutely continuous, and $\phi(t)$ and $t^s\phi^{(s)}(t)$ are bounded in $(1,\infty)$, then, for $s-1 < \lambda < s$, $\delta = \lambda-s+1$, $t \geqslant 1$,*

(i) $$\frac{1}{\Gamma(1-\delta)}\int\limits_t^\infty (v-t)^{-\delta}\phi^{(s)}(v)\,dv = \phi^{(\lambda)}(t),$$

(ii) *$\phi^{(\lambda)}(t)$ is absolutely continuous,*

(iii) $t^\lambda\phi^{(\lambda)}(t) = O(1)$.

Suppose that $x > t \geqslant 1$. Since $\phi'(t)$ is absolutely continuous, we have

$$\frac{d}{dt}\int\limits_t^x (u-t)^{-\delta}\phi(u)\,du = -\frac{d}{dt}\int\limits_t^x (u-t)^{-\delta}\,du\int\limits_u^x \phi'(v)\,dv + \phi(x)(x-t)^{-\delta}$$

$$= \int\limits_t^x (v-t)^{-\delta}\phi'(v)\,dv + \phi(x)(x-t)^{-\delta}.$$

Hence, since $\phi(x) = O(1)$ in $(1,\infty)$ and, by Lemma 2, $\phi'(v) = O(v^{-1})$ in $(1,\infty)$, we obtain

$$\phi^{(\delta)}(t) = \frac{1}{\Gamma(1-\delta)}\int\limits_t^\infty (v-t)^{-\delta}\phi'(v)\,dv, \tag{3.13}$$

which is conclusion (i) for the case $s = 1$.

Suppose now that $s \geqslant 2$. Then, by Lemma 2,

$$\phi'(v) = o(1), \qquad \phi''(v) = O(v^{-2})$$

as $v\to\infty$; and thus it follows from (3.13) that

$$\Gamma(1-\delta)\phi^{(\delta+1)}(t) = -\frac{d}{dt}\int\limits_t^\infty (v-t)^{-\delta}\,dv\int\limits_v^\infty \phi''(w)\,dw$$

$$= -\frac{d}{dt}\int\limits_t^\infty dv\int\limits_v^\infty (w-v)^{-\delta}\phi''(w)\,dw$$

$$= \int\limits_t^\infty (w-t)^{-\delta}\phi''(w)\,dw.$$

This proves result (i) for the case $s = 2$, and repetition of the argument yields the result when $s > 2$.

Since $\phi^{(s)}(v)$ is absolutely continuous and $v^s\phi^{(s)}(v) = O(1)$ in $(1,\infty)$, conclusions (ii) and (iii) follow directly from (i).

4. LEMMA 5. *For* $y > w > v > 0$, $0 < \delta < 1$,

$$\int_v^w (y-t)^{\delta-2}\,dt < (1-\delta)^{-\frac{1}{2}}(w-v)(y-w)^{\frac{1}{2}\delta-1}(y-v)^{\frac{1}{2}\delta-1}.$$

The result is obtained on taking the root of the product of the following two inequalities:

$$\int_v^w (y-t)^{\delta-2}\,dt < (y-v)^{\delta}\int_v^w (y-t)^{-2}\,dt = (w-v)(y-w)^{-1}(y-v)^{\delta-1};$$

$$\int_v^w (y-t)^{\delta-2}\,dt = (1-\delta)^{-1}\left\{y-v-(y-w)\left(\frac{y-v}{y-w}\right)^{\delta}\right\}(y-w)^{\delta-1}(y-v)^{-1}$$
$$< (1-\delta)^{-1}(w-v)(y-w)^{\delta-1}(y-v)^{-1}.$$

LEMMA 6. *For* $y > u > 1$, $0 < \delta < 1$,

$$\left|\int_1^u (y-t)^{\delta-2}\,dt\int_1^t (u-v)^{\delta-1}f(v)\,dv\right|$$
$$\leqslant (1-\delta)^{-\frac{1}{2}}\Gamma(\delta+1)(y-u)^{\frac{1}{2}\delta-1}\int_1^u (y-v)^{\frac{1}{2}\delta-1}|f_\delta(v)|\,dv.$$

It has been shown by M. Riesz* that, for $u > t > 1$,

$$\Gamma(1-\delta)\int_1^t (u-v)^{\delta-1}f(v)\,dv = \delta\int_1^t f_\delta(v)\,dv\int_t^u (u-w)^{\delta-1}(w-v)^{-\delta-1}\,dw.$$

In view of this and Lemma 5 we have

$$\Gamma(1-\delta)\left|\int_1^u (y-t)^{\delta-2}\,dt\int_1^t (u-v)^{\delta-1}f(v)\,dv\right|$$
$$\leqslant \delta\int_1^u (y-t)^{\delta-2}\,dt\int_1^t |f_\delta(v)|\,dv\int_t^u (u-w)^{\delta-1}(w-v)^{-\delta-1}\,dw$$
$$= \delta\int_1^u |f_\delta(v)|\,dv\int_v^u (u-w)^{\delta-1}(w-v)^{-\delta-1}\,dw\int_v^w (y-t)^{\delta-2}\,dt$$
$$\leqslant \delta(1-\delta)^{-\frac{1}{2}}\int_1^u (y-v)^{\frac{1}{2}\delta-1}|f_\delta(v)|\,dv\int_v^u (u-w)^{\delta-1}(w-v)^{-\delta}(y-w)^{\frac{1}{2}\delta-1}\,dw$$
$$\leqslant \delta(1-\delta)^{-\frac{1}{2}}(y-u)^{\frac{1}{2}\delta-1}\int_1^u (y-v)^{\frac{1}{2}\delta-1}|f_\delta(v)|\,dv\int_v^u (u-w)^{\delta-1}(w-v)^{-\delta}\,dw.$$

The result follows.

* M. Riesz, *Acta Univ. Hungaricae Szeged*, 1 (1923), 114–26. See also S. Verblunsky, *Proc. London Math. Soc.* (2) 32 (1931), 163–99.

5. LEMMA 7. *For* $\lambda > 0$, $\int_1^\infty f(t)\,dt$ *is summable* $|C,\lambda|$ *if and only if*

$$\int_1^\infty y^{-\lambda-1}|I_\lambda\{yf(y)\}|\,dy < \infty.$$

We have, for $y > 1$,

$$\frac{d}{dy}\left\{y^{-\lambda}\int_1^y (y-u)^{\lambda}f(u)\,du\right\}$$
$$= \lambda y^{-\lambda}\int_1^y (y-u)^{\lambda-1}f(u)\,du-\lambda y^{-\lambda-1}\int_1^y (y-u)^{\lambda}f(u)\,du$$
$$= \lambda y^{-\lambda-1}\int_1^y (y-u)^{\lambda-1}uf(u)\,du.$$

The result follows.

LEMMA 8. *If* $\int_1^\infty f(t)\,dt$ *is summable* $|C,\lambda|$, *where* $\lambda \geqslant 0$, *then*

$$tf(t) = o(1)|C,\lambda+1| \text{ as } t \to \infty.*$$

This follows from the identity

$$t^{-\lambda-1}\int_1^t (t-u)^{\lambda}uf(u)\,du = t^{-\lambda}\int_1^t (t-u)^{\lambda}f(u)\,du-t^{-\lambda-1}\int_1^t (t-u)^{\lambda+1}f(u)\,du.$$

LEMMA 9. *If* $\int_1^\infty f(t)\,dt$ *is summable* $|C,\lambda|$, *where* $\lambda \geqslant 1$, *then*

$$\int_1^\infty t^{-2}I_1\{tf(t)\}\,dt \text{ is summable } |C,\lambda-1|.$$

This is a simple special case of a result established elsewhere.†

6. Proof of Theorem 1

Version (*a*). We write, for $t \geqslant 1$,

$$g(t) = tf(t).$$

Case 1. Suppose that $0 < \lambda < 1$. In view of Lemma 7 we may replace hypothesis (i) by

$$\int_1^\infty y^{-\lambda-1}|g_\lambda(y)|\,dy < \infty; \qquad (6.1)$$

* i.e. $t^{-\lambda-1}I_{\lambda+1}\{tf(t)\}$ is of bounded variation in $(1,\infty)$ and is $o(1)$ as $t \to \infty$.

† D. Borwein (1), Theorem 1.

and the required conclusion by

$$\int_1^\infty y^{-\lambda-1}\,dy\left|\int_1^y (y-t)^{\lambda-1}g(t)\phi(t)\,dt\right| < \infty. \tag{6.2}$$

Write

$$P(u,y) = -\frac{\phi^{(\lambda)}(u)}{\Gamma(\lambda)}+\frac{\lambda}{\Gamma(\lambda)\Gamma(1-\lambda)}\int_{2y}^{\infty}(v-u)^{-\lambda-1}\phi(v)\,dv \quad (1 \leqslant u < 2y), \tag{6.3}$$

$$\begin{aligned} Q(u,y) &= \int_1^u (u-t)^{\lambda-1}(y-t)^{\lambda-1}g(t)\,dt \\ &= \Gamma(\lambda)(y-u)^{\lambda-1}g_\lambda(u)-(1-\lambda)\int_1^u (y-t)^{\lambda-2}\,dt\int_1^t (u-v)^{\lambda-1}g(v)\,dv \\ &\qquad (1 \leqslant u < y); \end{aligned} \tag{6.4}$$

$$M = \Gamma(\lambda)+(1-\lambda)^{\frac{1}{2}}\Gamma(\lambda+1)+\max_{u\geqslant 1}\left\{\frac{u^\lambda|\phi^{(\lambda)}(u)|}{\Gamma(\lambda)}+\frac{|\phi(u)|}{\Gamma(\lambda)\Gamma(1-\lambda)}\right\}, \tag{6.5}$$

where max denotes the essential upper bound. In view of hypotheses (ii) and (iv), M is finite.

Then, for $1 \leqslant u < 2y$,

$$|P(u,y)| \leqslant M\{u^{-\lambda}+(2y-u)^{-\lambda}\}, \tag{6.6}$$

and, by Lemma 6, for $1 \leqslant u < y$,

$$|Q(u,y)| \leqslant M(y-u)^{\lambda-1}|g_\lambda(u)|+M(y-u)^{\frac{1}{2}\lambda-1}\int_1^u (y-v)^{\frac{1}{2}\lambda-1}|g_\lambda(v)|\,dv. \tag{6.7}$$

For y such that

$$\int_1^y (y-u)^{\lambda-1}|g(u)|\,du < \infty \quad (y > 1), \tag{6.8}$$

we have, in virtue of Lemma 1, (6.3) and (6.6),

$$\begin{aligned} \int_1^y (y-t)^{\lambda-1}g(t)\phi(t)\,dt &= \int_1^y (y-t)^{\lambda-1}g(t)\,dt\int_t^{2y}(u-t)^{\lambda-1}P(u,y)\,du \\ &= \int_1^y P(u,y)\,du\int_1^u (y-t)^{\lambda-1}(u-t)^{\lambda-1}g(t)\,dt+ \\ &\quad +\int_y^{2y} P(u,y)\,du\int_1^y (y-t)^{\lambda-1}(u-t)^{\lambda-1}g(t)\,dt \\ &= \int_1^y P(u,y)Q(u,y)\,du+\int_y^{2y}P(u,y)Q(y,u)\,du; \end{aligned} \tag{6.9}$$

where, by (6.6) and (6.7),

$$\begin{aligned} &\left|\int_1^y P(u,y)Q(u,y)\,du\right| \\ &\leqslant 2M^2\int_1^y u^{-\lambda}\,du\left\{(y-u)^{\lambda-1}|g_\lambda(u)|+(y-u)^{\frac{1}{2}\lambda-1}\int_1^u (y-v)^{\frac{1}{2}\lambda-1}|g_\lambda(v)|\,dv\right\} \\ &\leqslant 2M^2\int_1^y u^{-\lambda}(y-u)^{\lambda-1}|g_\lambda(u)|\,du+ \\ &\qquad +2M^2\int_1^y (y-v)^{\frac{1}{2}\lambda-1}|g_\lambda(v)|\,dv\int_v^y v^{-\lambda}(y-u)^{\frac{1}{2}\lambda-1}\,du \\ &= 2M^2(1+2\lambda^{-1})\int_1^y u^{-\lambda}(y-u)^{\lambda-1}|g_\lambda(u)|\,du, \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} &\left|\int_y^{2y} P(u,y)Q(y,u)\,du\right| \\ &\leqslant 2M^2\int_y^{2y}(2y-u)^{-\lambda}\,du\left\{(u-y)^{\lambda-1}|g_\lambda(y)|+(u-y)^{\frac{1}{2}\lambda-1}\int_1^y (y-v)^{\frac{1}{2}\lambda-1}|g_\lambda(v)|\,dv\right\} \\ &= 2M^2|g_\lambda(y)|\int_y^{2y}(2y-u)^{-\lambda}(u-y)^{\lambda-1}\,du+ \\ &\qquad +2M^2\int_1^y (y-v)^{\frac{1}{2}\lambda-1}|g_\lambda(v)|\,dv\int_y^{2y}(2y-u)^{-\lambda}(u-y)^{\frac{1}{2}\lambda-1}\,du \\ &= 2M^2\mathrm{B}(\lambda,1-\lambda)|g_\lambda(y)|+2M^2\mathrm{B}(\tfrac{1}{2}\lambda,1-\lambda)y^{-\frac{1}{2}\lambda}\int_1^y (y-v)^{\frac{1}{2}\lambda-1}|g_\lambda(v)|\,dv. \end{aligned} \tag{6.11}$$

Let $N = 2M^2\{1+2\lambda^{-1}+\mathrm{B}(\lambda,1-\lambda)+\mathrm{B}(\frac{1}{2}\lambda,1-\lambda)\}$.

It is familiar that (6.8) holds for almost all y in $(1,\infty)$, and so it follows from (6.9), (6.10), and (6.11) that

$$\begin{aligned} &\int_1^\infty y^{-\lambda-1}\,dy\left|\int_1^y (y-t)^{\lambda-1}g(t)\phi(t)\,dt\right| \\ &\leqslant N\int_1^\infty u^{-\lambda}|g_\lambda(u)|\,du\int_u^\infty (y-u)^{\lambda-1}y^{-\lambda-1}\,dy+N\int_1^\infty y^{-\lambda-1}|g_\lambda(y)|\,dy+ \\ &\qquad +N\int_1^\infty |g_\lambda(v)|\,dv\int_v^\infty y^{-\frac{3}{2}\lambda-1}(y-v)^{\frac{1}{2}\lambda-1}\,dy \\ &= N\{\lambda^{-1}+1+\mathrm{B}(\tfrac{1}{2}\lambda,\lambda+1)\}\int_1^\infty y^{-\lambda-1}|g_\lambda(y)|\,dy. \end{aligned}$$

The result, (6.2), now follows from (6.1).

Since the theorem is trivially true when $\lambda = 0$, only the following case remains to be considered.

*Case 2.** *Suppose that* $\lambda \geqslant 1$, *and assume the theorem with* λ *replaced by* $\lambda-1$. In virtue of Lemma 1 (i), we may further suppose, without loss in generality, that $\phi(t)$ is absolutely continuous. Then, for $t > 1$,

$$\int_1^t f(u)\phi(u)\,du = \int_1^t u^{-1}g(u)\phi(u)\,du$$

$$= t^{-1}g_1(t)\phi(t)+\int_1^t u^{-2}g_1(u)\{\phi(u)-u\phi'(u)\}\,du. \qquad (6.12)$$

By Lemma 9, a consequence of hypothesis (i) is that

$$\int_1^\infty u^{-2}g_1(u)\,du \text{ is summable } |C,\lambda-1|. \qquad (6.13)$$

Let $s = [\lambda]$, $\delta = \lambda-s$. Then, by Lemma 2,

$$t^{\delta+r}\phi^{(\delta+r)}(t) = O(1) \text{ in } (1,\infty) \quad (r = 0,1,\ldots,s).$$

Hence, by conclusions (ii) and (iv) of Lemma 3,

$$t\phi'(t) = O(1) \text{ in } (1,\infty),$$

and, for $t \geqslant 1$,

$$t^{\lambda-1}D^{\lambda-1}\{t\phi'(t)\} = t^{\lambda-1}(d/dt)^{s-1}\{\delta\phi^{(\delta)}(t)+t\phi^{(\delta+1)}(t)\}$$

$$= (\lambda-1)t^{\lambda-1}\phi^{(\lambda-1)}(t)+t^{\lambda}\phi^{(\lambda)}(t).$$

Clearly then both $\phi(t)$ and $t\phi'(t)$ satisfy the hypotheses of $\phi(t)$ with λ replaced by $\lambda-1$: and so, in view of (6.13) and our assumption,

$$\int_1^\infty u^{-2}g_1(u)\phi(u)\,du \text{ is summable } |C,\lambda-1|, \qquad (6.14)$$

and

$$\int_1^\infty u^{-2}g_1(u)\{\phi(u)-u\phi'(u)\}\,du \text{ is summable } |C,\lambda-1|. \qquad (6.15)$$

It follows from (6.14), by Lemma 8, that

$$t^{-1}g_1(t)\phi(t) = o(1)|C,\lambda| \text{ as } t\to\infty. \qquad (6.16)$$

In view now of (6.12), (6.15), and (6.16),

$$\int_1^\infty f(u)\phi(u)\,du \text{ is summable } |C,\lambda|,$$

and Case 2 is thus proved by induction. This completes the proof of version (*a*) of the theorem.

* Cf. Bosanquet (**4**), 43.

Version (*b*) now follows from Lemmas 2 and 4. Though weaker than version (*a*), version (*b*) has been included since it involves only derivatives of integral order.

PART 2

7. In order to show that conditions (ii) and (iv) of Theorem 1 (*a*) cannot be relaxed we shall prove the following result.

THEOREM 2. *If* $\lambda \geqslant 0$, *and if*

(i) $\phi^{(\lambda)}(t)$ *is absolutely continuous,*

(ii) $\int_1^\infty f(t)\phi(t)\,dt$ *is bounded* (C)* *whenever* $\int_1^\infty f(t)\,dt$ *is summable* $|C,\lambda|$,

then, for $t \geqslant 1$, $\phi(t)$ *is essentially bounded and* $t^{\lambda}\phi^{(\lambda)}(t) = O(1)$.

We shall suppose, in what follows, that all functions discussed are real.† The following additional lemmas are required.

8. LEMMA 10. *If* $\int_1^\infty f(t)\phi(t)\,dt$ *is bounded* (C) *whenever* $\int_1^\infty |f(t)|\,dt < \infty$, *then* $\phi(t)$ *is essentially bounded in* $(1,\infty)$.

This can be established by a proof given elsewhere‡ for a slightly weaker form of the result.

LEMMA 11. *If* $\phi(t)$ *is continuous and unbounded in* $(1,\infty)$, *then, corresponding to any non-negative integer* s, *there is a function* $f(t)$ *such that* $f^{(s)}(t)$ *is absolutely continuous,* $f(1) = f'(1) = \ldots = f^{(s)}(1) = 0$,

$$\int_1^\infty |f(t)|\,dt < \infty, \quad \textit{and} \quad \int_1^\infty f(t)\phi(t)\,dt = \infty.$$

This has been established elsewhere.§

LEMMA 12. *If* $\phi(t)$ *is essentially bounded in* $(1,\infty)$, *and, for* $0 < \delta < 1$, $\phi^{(\delta)}(t)$ *is absolutely continuous, then, for* $x > 1$,

$$\int_1^x f(t)\phi(t)\,dt = -\int_1^x f_\delta(t)\phi^{(\delta)}(t)\,dt+\frac{\delta}{\Gamma(1-\delta)}\int_1^x f_\delta(t)\,dt\int_x^\infty (u-t)^{-\delta-1}\phi(u)\,du.$$

* i.e. $t^{-\mu}I_{\mu+1}\{f(t)\phi(t)\} = O(1)$ in $(1,\infty)$, for some non-negative μ.

† There is clearly no loss in generality if the theorem is proved for real functions $f(t)$ and $\phi(t)$.

‡ Borwein (**2**), Lemma 2.

§ Ibid. Lemma 1, Case 2.

In view of Lemma 1, we have

$$\int_1^x f(t)\phi(t)\,dt = -\frac{1}{\Gamma(\delta)}\int_1^x f(t)\,dt\int_t^x (u-t)^{\delta-1}\phi^{(\delta)}(u)\,du+$$

$$+\frac{\delta}{\Gamma(\delta)\Gamma(1-\delta)}\int_1^x f(t)\,dt\int_t^x (u-t)^{\delta-1}\,du\int_x^\infty (v-u)^{-\delta-1}\phi(v)\,dv$$

$$= -\frac{1}{\Gamma(\delta)}\int_1^x \phi^{(\delta)}(u)\,du\int_1^u (u-t)^{\delta-1}f(t)\,dt+$$

$$+\frac{\delta}{\Gamma(\delta)\Gamma(1-\delta)}\int_1^x du\int_1^u (u-t)^{\delta-1}f(t)\,dt\int_x^\infty (v-u)^{-\delta-1}\phi(v)\,dv;$$

the inversions being justified by the absolute convergence of the integrals concerned. The result follows.

Lemma 13. *If $f(t) = o(1)|C,\lambda|$ as $t\to\infty$, where $\lambda\geqslant 0$, then, for $p>-1$, $t^pf(t) = o(t^p)|C,\lambda|$ as $t\to\infty$.*

This is a special case of a result established elsewhere.*

Lemma 14. *If $\lambda>0$ and $\int_1^\infty t^{-\lambda}|f_\lambda(t)|\,dt<\infty$, then, for $\mu>0$ and*

$$\lambda\geqslant\nu>0,$$

(i) $\int_1^\infty t^{-\lambda-\mu}|f_{\lambda+\mu}(t)|\,dt<\infty$,

(ii) $\int_1^\infty t^{\nu-\lambda}f_{\lambda-\nu}(t)\,dt$ *is summable* $|C,\nu|$.

Result (i) is a consequence of the following inequality:†

$$\int_1^\infty t^{-\lambda-\mu}|f_{\lambda+\mu}(t)|\,dt \leqslant \frac{1}{\Gamma(\mu)}\int_1^\infty t^{-\lambda-\mu}\,dt\int_1^t (t-u)^{\mu-1}|f_\lambda(u)|\,du$$

$$=\frac{1}{\Gamma(\mu)}\int_1^\infty |f_\lambda(u)|\,du\int_u^\infty t^{-\lambda-\mu}(t-u)^{\mu-1}\,dt$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\lambda+\mu)}\int_1^\infty u^{-\lambda}|f_\lambda(u)|\,du.$$

Result (ii) is a special case of a theorem given elsewhere.‡

* Borwein (**1**), Lemma 4.
† Cf. L. S. Bosanquet (**3**).
‡ Borwein (**1**), Theorem 2.

9. Lemma 15. *If $\lambda-1$ is positive and non-integral, and if*

(i) $\int_1^\infty t^{-\lambda}|f_\lambda(t)|\,dt<\infty$,

(ii) *$\phi(t)$ is essentially bounded in $(1,\infty)$,*

(iii) *$\phi^{(\lambda-1)}(t)$ is absolutely continuous and*

$$t^{\lambda-1}\phi^{(\lambda-1)}(t) = O(1) \text{ in } (1,\infty),$$

then, for $\delta=\lambda-[\lambda]$, $s=[\lambda]+1$,

$$h(x) = \int_1^x f_\delta(t)\,dt\int_x^\infty (u-t)^{-\delta-1}\phi(u)\,du = O(1)\ (C,s) \text{ in } (1,\infty).*$$

If $0<\lambda<1$, the result follows from hypotheses (i) *and* (ii) *alone.*

Suppose that $y\geqslant x>1$, $r\geqslant 0$. Denote the finite essential upper bound of $|\phi(t)|$ in $(1,\infty)$ by M, and write

$$h(x,r) = \frac{1}{\Gamma(r+1)}\int_1^x (x-t)^r f_\delta(t)\,dt\int_x^\infty (u-t)^{-\delta-1}\phi(u)\,du,$$

$$k(x,r) = \int_1^x (x-t)^r|f_\delta(t)|\,dt\int_x^\infty (u-t)^{-\delta-1}|\phi(u)|\,du.$$

Then $h(x,0)=h(x)$, and

$$k(x,r)\leqslant\delta^{-1}M\int_1^x (x-t)^{r-\delta}|f_\delta(t)|\,dt$$

$$\leqslant\frac{M\Gamma(r+1-\delta)}{\delta\Gamma(r+1)}\int_1^x (x-t)^r|f(t)|\,dt.$$

Hence $$k(y,r)<\infty \quad\text{and}\quad \int_1^y k(x,r)\,dx<\infty; \tag{9.1}$$

and so

$$\Gamma(r+1)\int_1^y h(x,r)\,dx = \int_1^y dx\int_x^\infty \phi(u)\,du\int_1^x (u-t)^{-\delta-1}(x-t)^r f_\delta(t)\,dt$$

$$=\int_1^y dx\int_x^y \phi(u)\,du\int_1^x (u-t)^{-\delta-1}(x-t)^r f_\delta(t)\,dt+$$

$$+\int_1^y dx\int_y^\infty \phi(u)\,du\int_1^x (u-t)^{-\delta-1}(x-t)^r f_\delta(t)\,dt$$

* i.e. $h_s(x) = O(x^s)$ in $(1,\infty)$.

$$= \int_1^y \phi(u)\,du \int_1^u dx \int_1^x (u-t)^{-\delta-1}(x-t)^r f_\delta(t)\,dt +$$

$$+ \int_y^\infty \phi(u)\,du \int_1^y dx \int_1^x (u-t)^{-\delta-1}(x-t)^r f_\delta(t)\,dt$$

$$= \frac{1}{r+1}\int_1^y \phi(u)\,du \int_1^u (u-t)^{r-\delta} f_\delta(t)\,dt +$$

$$+\frac{1}{r+1}\int_y^\infty \phi(u)\,du \int_1^y (u-t)^{-\delta-1}(y-t)^{r+1} f_\delta(t)\,dt$$

$$= \frac{\Gamma(r+1-\delta)}{r+1}\int_1^y f_{r+1}(u)\phi(u)\,du + \Gamma(r+1)h(y,r+1).$$

Consequently

$$h_s(x) = h(x,s) + \sum_{r=0}^{s-1} \frac{\Gamma(r+1-\delta)}{(r+1)!} I_{s-r}\{f_{r+1}(x)\phi(x)\}. \tag{9.2}$$

Further it follows from hypothesis (i), by Lemma 14 (i), that

$$\int_1^\infty t^{-s}|f_s(t)|\,dt < \infty,$$

and so, in view of hypothesis (ii),

$$\int_1^\infty t^{-s}|f_s(t)\phi(t)|\,dt < \infty. \tag{9.3}$$

When $s \geqslant 2$ and $r = 0, 1, \dots, s-2$, we have, in virtue of hypothesis (i) and Lemma 14 (ii), that

$$\int_1^\infty t^{-r-1} f_{r+1}(t)\,dt \text{ is summable } |C, \lambda-r-1|;$$

and thus, since we can deduce from hypotheses (ii) and (iii) and Lemma 2 that $\phi(t)$ satisfies conditions (ii), (iii) and (iv) of Theorem 1 (a), with λ replaced by $\lambda-r-1$,

$$\int_1^\infty t^{-r-1} f_{r+1}(t)\phi(t)\,dt \text{ is summable } |C, \lambda-r-1|. \tag{9.4}$$

It follows now from (9.3) and (9.4), by Lemma 8, that, for

$$r = 0, 1, \dots, s-1 \quad (s \geqslant 1),$$

$$x^{-r} f_{r+1}(x)\phi(x) = o(1)\ |C, s-r| \text{ as } x \to \infty;$$

and thus, by Lemma 13,

$$f_{r+1}(x)\phi(x) = o(x^r)\ |C, s-r| \text{ as } x \to \infty.$$

Hence
$$\sum_{r=0}^{s-1} \frac{\Gamma(r+1-\delta)}{(r+1)!} I_{s-r}\{f_{r+1}(x)\phi(x)\} = O(x^s) \text{ in } (1,\infty). \tag{9.5}$$

We prove next that

$$h(x,s) = O(x^s) \text{ in } (1,\infty). \tag{9.6}$$

In view of the first inequality in (9.1), we have

$$s!\,h(x,s) = \int_x^\infty \phi(u)\,du \int_1^x (u-t)^{-\delta-1}(x-t)^s f_\delta(t)\,dt$$

$$= (-1)^{s-1}\int_x^\infty \phi(u)\,du \int_1^x f_\lambda(t)\,(d/dt)^{s-1}\{(u-t)^{-\delta-1}(x-t)^s\}\,dt$$

$$= \sum_{r=0}^{s-1} c_r X_r, \tag{9.7}$$

where $c_0, c_1, \dots, c_{s-1}$, are constants and

$$|X_r| = \left|\int_x^\infty \phi(u)\,du \int_1^x (u-t)^{-\delta-1-r}(x-t)^{r+1} f_\lambda(t)\,dt\right|$$

$$\leqslant M\int_x^\infty du \int_1^x (u-t)^{-\delta-1-r}(x-t)^{r+1}|f_\lambda(t)|\,dt$$

$$= \frac{M}{\delta+r}\int_1^x (x-t)^{1-\delta}|f_\lambda(t)|\,dt$$

$$\leqslant \delta^{-1}Mx^{1-\delta+\lambda}\int_1^x (1-t/x)^{1-\delta}t^{-\lambda}|f_\lambda(t)|\,dt$$

$$\leqslant \delta^{-1}Mx^s\int_1^\infty t^{-\lambda}|f_\lambda(t)|\,dt.$$

Thus, in virtue of hypothesis (i), (9.6) follows from (9.7).

The result is now a consequence of (9.2), (9.5), and (9.6).*

10. Proof of Theorem 2

We deduce from hypothesis (ii) and Lemma 10 that

$$\phi(t) \text{ is essentially bounded in } (1,\infty). \tag{10.1}$$

We have thus only to prove that

$$t^\lambda \phi^{(\lambda)}(t) = O(1) \text{ in } (1,\infty). \tag{10.2}$$

* Clearly only hypotheses (i) and (ii) are used when $0 < \lambda < 1$.

Case 1. Suppose that $0<\lambda<1$, *and assume that*

$$t^{\lambda}\phi^{(\lambda)}(t) \text{ is unbounded in } (1,\infty). \tag{10.3}$$

It follows from the assumption, by Lemma 11, with $\phi(t)$ replaced by $t^{\lambda}\phi^{(\lambda)}(t)$, that there is an absolutely continuous function $g(t)$* such that $g(1)=0$,

$$\int_1^\infty t^{-\lambda}|g(t)|\,dt<\infty \tag{10.4}$$

and

$$\int_1^\infty g(t)\phi^{(\lambda)}(t)\,dt=\infty. \tag{10.5}$$

Now define, for almost all $t\geqslant 1$,

$$f(t)=I_{1-\lambda}g'(t).$$

Clearly then, for $t\geqslant 1$,

$$f_\lambda(t)=g(t). \tag{10.6}$$

It follows from (10.4) and (10.6), by Lemma 14 (ii), that

$$\int_1^\infty f(t)\,dt \text{ is summable } |C,\lambda|;$$

and thus, by hypothesis (ii),

$$\int_1^\infty f(t)\phi(t)\,dt \text{ is bounded } (C). \tag{10.7}$$

On the other hand, in virtue of (10.6) and Lemma 12, we have, for $x>1$,

$$\int_1^x f(t)\phi(t)\,dt$$

$$=-\int_1^x g(t)\phi^{(\lambda)}(t)\,dt+\frac{\lambda}{\Gamma(1-\lambda)}\int_1^x f_\lambda(t)\,dt\int_x^\infty (u-t)^{-\lambda-1}\phi(u)\,du. \tag{10.8}$$

Now it follows from (10.1), (10.4) and (10.6), by Lemma 15, that the repeated integral in (10.8) is $O(1)$ $(C,1)$ in $(1,\infty)$. We thus deduce from (10.5) and (10.8) that, in contradiction to (10.7),

$$\int_1^\infty f(t)\phi(t)\,dt \text{ is not bounded } (C).$$

Therefore the assumption is false, and so for this case (10.2) holds.

Case 2. Suppose that $\lambda>1$ *and* $\lambda\neq 2,3,\ldots$. Let

$$\delta=\lambda-[\lambda] \quad\text{and}\quad s=[\lambda]+1.$$

* $t^{-\lambda}g(t)$ is the function $f(t)$ of Lemma 11.

Since $\phi(t)$ satisfies the hypotheses of the theorem, with λ replaced by δ, it follows from Case 1 that

$$t^{\delta}\phi^{(\delta)}(t)=O(1) \text{ in } (1,\infty). \tag{10.9}$$

Assume now that

$$t^{\lambda-1}\phi^{(\lambda-1)}(t)=O(1) \text{ in } (1,\infty) \tag{10.10}$$

and

$$t^{\lambda}\phi^{(\lambda)}(t) \text{ is unbounded in } (1,\infty). \tag{10.11}$$

As in Case 1, it follows from (10.11), by Lemma 11, that there is a function $g(t)$ such that

$$g^{(s-1)}(t) \text{ is absolutely continuous,} \tag{10.12}$$

$$g(1)=g'(1)=\ldots=g^{(s-1)}(1)=0, \tag{10.13}$$

$$\int_1^\infty t^{-\lambda}|g(t)|\,dt<\infty \tag{10.14}$$

and

$$\int_1^\infty g(t)\phi^{(\lambda)}(t)=\infty. \tag{10.15}$$

Now define, for almost all $t\geqslant 1$,

$$f(t)=I_{s-\lambda}g^{(s)}(t).$$

Then, in view of (10.12) and (10.13), we have, for $t\geqslant 1$,

$$f_\lambda(t)=g(t). \tag{10.16}$$

It follows, from (10.14) and (10.16), by Lemma 14 (ii), that

$$\int_1^\infty f(t)\,dt \text{ is summable } |C,\lambda|;$$

and thus, by hypothesis (ii),

$$\int_1^\infty f(t)\phi(t)\,dt \text{ is bounded } (C). \tag{10.17}$$

On the other hand, by Lemma 12, we have, for $x>1$,

$$\int_1^x f(t)\phi(t)\,dt-\frac{\delta}{\Gamma(1-\delta)}\int_1^x f_\delta(t)\,dt\int_x^\infty (u-t)^{-\delta-1}\phi(u)\,du$$

$$=-\int_1^x f_\delta(t)\phi^{(\delta)}(t)\,dt$$

$$=\sum_{r=1}^{s-1}(-1)^r f_{\delta+r}(x)\phi^{(\delta+r-1)}(x)+(-1)^s\int_1^x f_\lambda(t)\phi^{(\lambda)}(t)\,dt. \tag{10.18}$$

Now it follows from (10.14) and (10.16), by Lemma 14 (ii), that, for $r = 1, 2, \ldots, s-1$,

$$\int_1^\infty t^{-\delta-r} f_{\delta+r}(t)\, dt \text{ is summable } |C, s-r-1|.$$

Also, for such r, we can deduce from hypothesis (i), (10.1), (10.10) and Lemma 2 that $t^{\delta+r-1}\phi^{(\delta+r-1)}(t)$ satisfies the hypotheses of $\phi(t)$ in Theorem 1 (a), with λ replaced by $s-r-1$; and thus

$$\int_1^\infty t^{-\delta-r} f_{\delta+r}(t) \,.\, t^{\delta+r-1}\phi^{(\delta+r-1)}(t)\, dt \text{ is summable } |C, s-r-1|.$$

Hence, by Lemma 8, for $r = 1, 2, \ldots, s-1$,

$$f_{\delta+r}(x)\phi^{(\delta+r-1)}(x) = o(1)\ |C, s-r| \text{ as } x \to \infty. \tag{10.19}$$

Further, in view of hypothesis (i), (10.1), (10.10), (10.14), (10.16) and Lemma 15, the repeated integral in (10.18) is $O(1)$ (C, s) in $(1, \infty)$. Thus it follows from (10.15), (10.16), (10.18) and (10.19) that, in contradiction to (10.17),

$$\int_1^\infty f(t)\phi(t)\, dt \text{ is not bounded } (C).$$

Therefore the assumption is false; and thus, since $\phi(t)$ satisfies the hypotheses of the theorem with λ replaced by $\delta+r$ $(r = 1, 2, \ldots, s-1)$,

if $$t^{\delta+r-1}\phi^{(\delta+r-1)}(t) = O(1) \textit{ in } (1, \infty),$$

then $$t^{\delta+r}\phi^{(\delta+r)}(t) = O(1) \textit{ in } (1, \infty) \quad (r = 1, 2, \ldots, s-1).$$

Consequently (10.2) follows from (10.9).

Case 3. Suppose finally that λ is a non-negative integer. When $\lambda = 0$, (10.2) follows immediately from (10.1); and when $\lambda \geqslant 1$ we may argue as in Case 2, putting $\delta = 0$, $s = \lambda+1$, and omitting the repeated integral from (10.18).

This completes the proof of Theorem 2.

In conclusion, I should like to express my thanks to Dr. L. S. Bosanquet for advice and helpful criticism.

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