# ON ABSOLUTE RIESZ SUMMABILITY FACTORS

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1. Suppose throughout that $a$, $k$ are positive numbers, and that $p$ is the integer such that $k-1 < p \leqslant k$. Suppose that $\phi(w)$, $\psi(w)$ are functions with absolutely continuous $p$-th derivatives in every interval $[a, W]$, and that $\phi(w)$ is positive and unboundedly increasing. Let $\lambda = \{\lambda_n\}$ be an unboundedly increasing sequence with $\lambda_1 > 0$.

Given a series, $\sum_{n=1}^{\infty} a_n$, and a number, $m \geqslant 0$, we write

$$A_m(w) = \begin{cases} \sum_{\lambda_n \leqslant w} (w-\lambda_n)^m a_n & \text{if } w > \lambda_1, \\ 0 & \text{otherwise,} \end{cases}$$

and $A(w) = A_0(w)$.

If $w^{-m} A_m(w)$ tends to a finite limit as $w$ tends to infinity, the series, $\sum_{n=1}^{\infty} a_n$, is said to be summable $(R, \lambda, m)$; and it is said to be absolutely summable $(R, \lambda, m)$, or summable $|R, \lambda, m|$, if $w^{-m} A_m(w)$ is of bounded variation in the range $w \geqslant 0$.

We shall use the notation, ${}_wD_t^k f(t)$, to denote

$$\frac{(-1)^{p+1}}{\Gamma(p+1-k)} \left(\frac{\partial}{\partial t}\right)^{p+1} \int_t^w (u-t)^{p-k} f(u)\, du,$$

provided this expression is defined.

The object of this note is to obtain manageable conditions sufficient to ensure, when $k$ is not an integer, the truth of the proposition

$P$: $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ *is summable* $|R, \phi(\lambda), k|$ *whenever* $\sum_{n=1}^{\infty} a_n$ *is summable* $|R, \lambda, k|$.

The following theorems are known.

*For all* $k$:

T$_1$. *If* $\phi(w) = e^w$ *and* $\psi(w) = w^{-k}$, *then* $P$.

*For integral values of* $k$:

T$_2$. *If* (i) $\phi(w)$ *is a logarithmico-exponential function,*†

(ii) $\dfrac{1}{w} \leqslant \dfrac{\phi'(w)}{\phi(w)}$,

(iii) $\psi(w) = \left\{\dfrac{\phi(w)}{w\phi'(w)}\right\}^k$,

*then* $P$, *and*

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† For definitions and properties of logarithmico-exponential functions, see [**6**].

$T_3$. *If there is a function, $\gamma(w)$, defined and positive for $w \geqslant a$, such that*

(i) $\gamma(w) = O(w)$ *for* $w \geqslant a$,

(ii) $w^n \psi^{(n)}(w) = O\left(\left\{\frac{\gamma(w)}{w}\right\}^{k-n}\right)$ *for* $n = 0, 1, \ldots, k$ *and* $w \geqslant a$,

(iii) $\{\gamma(w)\}^n \phi^{(n)}(w) = O\{\phi(w)\}$ *for* $n = 1, 2, \ldots, k$ *and* $w \geqslant a$,

*then P.*

*For non-integral values of k:*

$T_4$. *If there is a function, $\gamma(w)$, defined and positive for $w \geqslant a$, such that*

(i) $\gamma(w) = O(w)$ *for* $w \geqslant a$,

(ii) $w^n \psi^{(n)}(w) = O\left(\left\{\frac{\gamma(w)}{w}\right\}^{k-n}\right)$ *for* $n = 0, 1, \ldots, p$ *and* $w \geqslant a$,

(iii) $\gamma(w)\phi'(w) = O\{\phi(w)\}$ *for* $w \geqslant a$ *or* $\{\gamma(w)\}^n \phi^{(n)}(w) = O\{\phi(w)\}$ *for* $n = 1, 2, \ldots, p$ *and* $w \geqslant a$, *according as* $0 < k < 1$ *or* $k > 1$,

(iv) *for* $t \geqslant a$,

$$w^k\, {}_wD_t^k\left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right)$$

*is of uniformly bounded variation with respect to w in the range* $[t, \infty)$,

*then P.*

$T_1$ is due to Tatchell [**7**], $T_2$ to Guha [**4**], and $T_3$ and $T_4$ to Dikshit [**2**, **3**].

Suppose, from now on, that $k$ is not an integer. Our main theorem is:

$T_A$. *If there is a function, $\gamma(w)$, defined and positive for $w \geqslant a$, such that*

(i) $\gamma(w) = O(w)$ *for* $w \geqslant a$,

(ii) (a) $\psi(w) = O\left(\left\{\frac{\gamma(w)}{w}\right\}^k\right)$ *for* $w \geqslant a$,

(b) $w^n \psi^{(n)}(w) = O\left(\left\{\frac{\gamma(w)}{w}\right\}^{p+1-n}\right)$ *for* $n = 1, 2, \ldots, p+1$ *and* $w \geqslant a$,

(iii) (a) $\{\gamma(w)\}^n \phi^{(n)}(w) = O\{\phi(w)\}$ *for* $n = 1, 2, \ldots, p+1$ *and* $w \geqslant a$,

(b) *for* $n = 2, 3, \ldots, p+1$,

$$\left\{\frac{\phi(w)}{\phi'(w)}\right\}^{n-1} \cdot \left\{\frac{\phi^{(n)}(w)}{\phi'(w)}\right\}$$

*is of bounded variation in the range* $w \geqslant a$,†

† This condition is void if $k < 1$.



(iv) $\left\{\frac{w\phi'(w)}{\phi(w)}\right\}^k \psi(w)$ *is of bounded variation in the range* $w \geqslant a$,

(v) $\frac{w\phi''(w)}{\phi'(w)}$ *and* $\frac{w\phi'(w)}{\phi(w)}$ *are non-negative monotonic non-decreasing for* $w \geqslant a$,

*then P.*

We shall, in fact, deduce $T_A$ from the more general theorem:

$T_B$. *If condition* (v) *of* $T_A$ *is replaced by*

(v)′ *for* $0 < v < 1$, $t \geqslant a$ *and* $\epsilon > 0$,

$$\left\{\frac{x(1-v)\phi'(t+vx)}{\phi(t+x) - \phi(t+vx)}\right\}^{p+1-k} \text{ and } \left\{\frac{\phi(t+vx)}{\phi(t+x)}\right\}^{k-p}$$

*are of uniformly bounded variation with respect to x in the range* $[\epsilon, \infty)$, *the other conditions of* $T_A$ *remaining unchanged,*

*then P.*

Evidently, $T_A$ includes $T_1$, with $k$ non-integral. A simple consequence of $T_A$, with $\gamma(w) = \frac{\phi(w)}{\phi'(w)}$, is that, for non-integral $k$, $P$ holds with

$$\psi(w) = \left\{\frac{\phi(w)}{w\phi'(w)}\right\}^k$$

provided $\phi(w)$ is a logarithmico-exponential function satisfying $\frac{1}{w} \prec \frac{\phi'(w)}{\phi(w)}$. Also, $T_A$ implies that $P$ is true when $\phi(w) = w$ and $\psi(w)$ is a logarithmico-exponential function tending to a non-zero finite limit. Using now a result due to Chandrasekharan [**1**], we can easily deduce that $T_2$ is also true when $k$ is non-integral. We have not investigated the exact relation between our theorems and $T_4$, but each condition of $T_A$ is simple to verify in particular cases, whereas the unwieldy condition (iv) of $T_4$ is not.

2. The following lemmas are required.

LEMMA 1. *The n-th derivative of $\{f(x)\}^m$ is a sum of constant multiples of a finite number of terms like*

$$\{f(x)\}^{m-\mu} \prod_{\nu=1}^{n} \{f^{(\nu)}(x)\}^{\alpha_\nu}$$

*where* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are non-negative integers such that*

$$1 \leqslant \sum_{\nu=1}^{n} \alpha_\nu = \mu \leqslant \sum_{\nu=1}^{n} \nu\alpha_\nu = n.$$

This is a particular case of a theorem due to Faa di Bruno. See [**8**; I, pp. 89–90].

LEMMA 2. *For* $w > t \geqslant 0$

$$
{}_wD_t^k\left\{1-\frac{\phi(t)}{\phi(w)}\right\}^k\psi(t)
$$

$$
=\frac{(-1)^{p+1}}{\Gamma(p+1-k)}\int_t^w (u-t)^{p-k}\left(\frac{\partial}{\partial u}\right)^{p+1}\left(\left\{1-\frac{\phi(u)}{\phi(w)}\right\}^k\psi(u)\right)du.
$$

This is similar to Lemma 5 (first part of proof) in [5].

LEMMA 3. *For* $w > 0$

$$
\int_0^w\left\{1-\frac{\phi(t)}{\phi(w)}\right\}^k\psi(t)\,dA(t)
$$

$$
=\frac{1}{\Gamma(k+1)}\int_0^w {}_wD_t^k\left(\left\{1-\frac{\phi(t)}{\phi(w)}\right\}^k\psi(t)\right)\frac{dA_k(t)}{dt}\,dt.
$$

This is similar to Lemma 6 in [5].

LEMMA 4.†

(i) *If* $G_1(x)=\int_a^x f_1(x,u)\,g_1(u)\,du$ *and* $f_1(x,x)=0$ *for* $x \geqslant a$, *then*

$$
\int_a^\infty |dG_1(x)| \leqslant \underset{u>a}{\overline{\mathrm{bd}}}\int_u^\infty |d_xf_1(x,u)|\,.\int_a^\infty |g_1(u)|\,du.
$$

(ii) *If* $G_2(x)=\int_{a/x}^1 f_2(x,u)\,g_2(u)\,du$ *and* $f_2\left(x,\frac{a}{x}\right)=0$ *for* $x \geqslant a$, *then*

$$
\int_a^\infty |dG_2(x)| \leqslant \underset{0<u<1}{\overline{\mathrm{bd}}}\int_{a/u}^\infty |d_xf_2(x,u)|\,.\int_0^1 |g_2(u)|\,du.
$$

(iii) *If* $G_3(x)=\int_0^1 f_3(x,u)\,g_3(u)\,du$, *then*

$$
\int_a^\infty |dG_3(x)| \leqslant \underset{0<u<1}{\overline{\mathrm{bd}}}\int_a^\infty |d_xf_3(x,u)|\,.\int_0^1 |g_3(u)|\,du.
$$

This is essentially the same as Lemma 1 in [7].

LEMMA 5. *For* $\mu = 0, 1, \ldots, p$ *and* $t \geqslant a$

$$
\Phi_\mu(w,t)=\{\phi(w)\}^{-k}\{\phi(w)-\phi(t)\}^{k-\mu}\{\phi(t)\}^\mu
$$

*is of uniformly bounded variation with respect to* $w$ *in the range* $[t,\infty)$.

The result is trivial if $\mu = 0$. For $\mu > 0$, $\Phi_\mu(w,t)$ is non-negative monotonic non-decreasing until $\mu\phi(t)=k\phi(w)$, and then non-negative monotonic non-increasing. Hence, the total variation of $\Phi_\mu(w,t)$ is at most $2(k-\mu)^{k-\mu}k^{-k}\mu^\mu$ which is constant, and independent of $t$.

† It is to be assumed that all integrals in this Lemma are defined in Lebesgue and Riemann-Stieltjes senses as appropriate, and that $G_1(a) = G_2(a) = 0$.



LEMMA 6.

(i) *If* $\dfrac{u\phi''(u)}{\phi'(u)}$ *is non-negative monotonic non-decreasing for* $u \geqslant a$, *then* $\dfrac{x(1-v)\phi'(t+vx)}{\phi(t+x)-\phi(t+vx)}$ *is a uniformly bounded non-increasing function of* $x$ *in the range* $(0,\infty)$ *for* $0<v<1$ *and* $t \geqslant a$.

(ii) *If* $\dfrac{u\phi'(u)}{\phi(u)}$ *is non-negative monotonic non-decreasing for* $u \geqslant a$, *then* $\dfrac{\phi(t+vx)}{\phi(t+x)}$ *is a uniformly bounded non-increasing function of* $x$ *in the range* $[0,\infty)$ *for* $0<v<1$ *and* $t \geqslant a$.

*Proof.* (i) Denoting $\dfrac{x(1-v)\phi'(t+vx)}{\phi(t+x)-\phi(t+vx)}$ by $F_1(x)$, we have, for $x>0$, $t \geqslant a$ and $0<v<1$,

$$
\frac{F_1'(x)}{F_1(x)}=\frac{v\phi''(t+vx)}{\phi'(t+vx)}+\frac{1}{x}-\frac{\phi'(t+x)-v\phi'(t+vx)}{\phi(t+x)-\phi(t+vx)}
$$

$$
=\frac{v\phi''(t+vx)}{\phi'(t+vx)}+\frac{F_2(x)-F_2(vx)}{x\{\phi(t+x)-\phi(t+vx)\}}
$$

where

$$
F_2(x)=\phi(t+x)-x\phi'(t+x),
$$

and so

$$
F_2'(x)=-x\phi''(t+x).
$$

Hence

$$
\frac{F_1'(x)}{F_1(x)}=\frac{v\phi''(t+vx)}{\phi'(t+vx)}+\frac{F_2'(\theta x)}{x\phi'(t+\theta x)} \qquad \text{where } v<\theta<1
$$

$$
=\frac{v\phi''(t+vx)}{\phi'(t+vx)}-\frac{\theta\phi''(t+\theta x)}{\phi'(t+\theta x)}
$$

$$
=\frac{1}{x}\left\{\frac{(t+vx)\phi''(t+vx)}{\phi'(t+vx)}\,\frac{vx}{t+vx}-\frac{(t+\theta x)\phi''(t+\theta x)}{\phi'(t+\theta x)}\,\frac{\theta x}{t+\theta x}\right\}
$$

$$
\leqslant 0,
$$

since $\dfrac{u\phi''(u)}{\phi'(u)}$ and $\dfrac{u}{t+u}$ are non-negative monotonic non-decreasing functions of $u$ for $u \geqslant a$. Since $F_1(x)$ is non-negative, and $\lim_{x\to 0+}F_1(x)=1$, the result follows.

(ii) Denoting $\dfrac{\phi(t+vx)}{\phi(t+x)}$ by $F_3(x)$, we have, for $x \geqslant 0$, $t \geqslant a$ and $0<v<1$,

$$\frac{F_3'(x)}{F_3(x)} = \frac{v\phi'(t+vx)}{\phi(t+vx)} - \frac{\phi'(t+x)}{\phi(t+x)}$$

$$= \frac{1}{x}\left\{\frac{(t+vx)\phi'(t+vx)}{\phi(t+vx)}\,\frac{vx}{t+vx} - \frac{(t+x)\phi'(t+x)}{\phi(t+x)}\,\frac{x}{t+x}\right\}$$

$$\leqslant 0,$$

since $\dfrac{u\phi'(u)}{\phi(u)}$ and $\dfrac{u}{t+u}$ are non-negative monotonic non-decreasing functions of $u$ for $u \geqslant a$. Since $F_3(x)$ is non-negative and $F_3(0) = 1$, the result follows.

3. *Proof of* $\mathrm{T}_B$. We assume, without loss of generality, that $A(w) = 0$ for $0 \leqslant w \leqslant a$ and that $w^{-k}A_k(w)$ is of bounded variation in the range $w \geqslant a$, so that

$$\int_a^\infty \left|\frac{d}{dt}\{t^{-k}A_k(t)\}\right| dt < \infty \tag{1}$$

and note that it is sufficient to prove that

$$\sum_{\phi(\lambda_n)\leqslant\phi(w)} \left\{1-\frac{\phi(\lambda_n)}{\phi(w)}\right\}^k \psi(\lambda_n)\,a_n,$$

which, for $w \geqslant a$, is equal to

$$\int_a^w \left\{1-\frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\,dA(t), \tag{2}$$

is of bounded variation in the range $w \geqslant a$.

In view of Lemma 3, we can express (2) in the form

$$\frac{1}{\Gamma(k+1)}\int_a^w {}_wD_t^k\left(\left\{1-\frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right)\frac{d}{dt}\{A_k(t)\}\,dt = \frac{1}{\Gamma(k)}\,I_1 + \frac{1}{\Gamma(k+1)}I_2,$$

where

$$I_1 = \int_a^w {}_wD_t^k\left(\left\{1-\frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right) t^{k-1}\frac{A_k(t)}{t^k}\,dt,$$

and

$$I_2 = \int_a^w {}_wD_t^k\left(\left\{1-\frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right) t^k\frac{d}{dt}\left\{\frac{A_k(t)}{t^k}\right\} dt.$$

By partial integration,

$$I_1 = \int_a^w q_1(w,t)\,\frac{d}{dt}\left\{\frac{A_k(t)}{t^k}\right\} dt$$

where

$$q_1(w,t) = \int_t^w {}_wD_x^k\left(\left\{1-\frac{\phi(x)}{\phi(w)}\right\}^k \psi(x)\right) x^{k-1}\,dx.$$



In order to show that $I_1$ is of bounded variation with respect to $w$ in the range $[a, \infty)$, it is sufficient, in view of (1) and Lemma 4(i), to prove that

$$\int_t^\infty |d_w q_1(w,t)| = O(1) \text{ for } t \geqslant a.$$

Now, by Lemma 2, for $w \geqslant t \geqslant a$,

$$(-1)^{p+1}\Gamma(p+1-k)\,q_1(w,t)$$

$$= \int_t^w x^{k-1}\,dx \int_x^w (u-x)^{p-k}\left(\frac{\partial}{\partial u}\right)^{p+1}\left(\left\{1-\frac{\phi(u)}{\phi(w)}\right\}^k \psi(u)\right) du$$

$$= \int_t^w \left(\frac{\partial}{\partial u}\right)^{p+1}\left(\left\{1-\frac{\phi(u)}{\phi(w)}\right\}^k \psi(u)\right) du \int_t^u (u-x)^{p-k} x^{k-1}\,dx$$

$$= \int_t^w u^p\left(\frac{\partial}{\partial u}\right)^{p+1}\left(\left\{1-\frac{\phi(u)}{\phi(w)}\right\}^k \psi(u)\right) du \int_{t/u}^1 (1-v)^{p-k} v^{k-1}\,dv$$

$$= \int_{t/w}^1 (1-v)^{p-k} v^{k-1}\,dv \int_{t/v}^w u^p\left(\frac{\partial}{\partial u}\right)^{p+1}\left(\left\{1-\frac{\phi(u)}{\phi(w)}\right\}^k \psi(u)\right) du$$

$$= \int_{t/w}^1 (1-v)^{p-k} v^{k-1}\,q_2\left(w, \frac{t}{v}\right) dv$$

where

$$q_2(w,y) = \int_y^w u^p\left(\frac{\partial}{\partial u}\right)^{p+1}\left(\left\{1-\frac{\phi(u)}{\phi(w)}\right\}^k \psi(u)\right) du.$$

Since

$$\int_0^1 (1-v)^{p-k} v^{k-1}\,dv = \frac{\Gamma(p+1-k)\,\Gamma(k)}{\Gamma(p+1)},$$

in order to prove that $I_1$ is of bounded variation with respect to $w$ in the range $[a, \infty)$, it is sufficient, in view of Lemma 4 (ii), to prove that

$$\int_y^\infty |d_w q_2(w,y)| = O(1) \text{ for } y \geqslant a.$$

Integration by parts yields, for $w \geqslant y \geqslant a$

$$q_2(w,y) = b_0\psi(y)\left\{1-\frac{\phi(y)}{\phi(w)}\right\}^k + \sum_{r=1}^p b_r y^r\left(\frac{\partial}{\partial y}\right)^r\left(\left\{1-\frac{\phi(y)}{\phi(w)}\right\}^k \psi(y)\right),$$

where $b_0, b_1, \ldots, b_p$ are constants.

By $\mathrm{T}_A$ (i) and (ii), $\psi(y) = O(1)$ for $y \geqslant a$, and hence the term with coefficient $b_0$ is of uniformly bounded variation with respect to $w$ in the range $[y, \infty)$ for $y \geqslant a$. Also, by Lemma 1 and Leibnitz's theorem on the differentiation of a product, the other terms can be expressed as sums of constant multiples of terms like

$$q_3(w,y) = \Phi_\mu(w,y)\,y^r\,\psi^{(r-n)}(y)\,\{\phi(y)\}^{-\mu}\prod_{\nu=1}^n \{\phi^{(\nu)}(y)\}^{\alpha_\nu}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative integers such that

$$0 \leqslant \sum_{\nu=1}^{n} \alpha_\nu = \mu \leqslant \sum_{\nu=1}^{n} \nu\alpha_\nu = n \leqslant r \leqslant p, \dagger$$

and

$$\Phi_\mu(w, y) = \{\phi(w)\}^{-k}\{\phi(w)-\phi(y)\}^{k-\mu}\{\phi(y)\}^{\mu}.$$

Now, in view of Lemma 5, $q_3(w, y)$ is of uniformly bounded variation with respect to $w$ in the range $[y, \infty)$ for $y \geqslant a$, since, if we take $m = p+1$ when $r > n$ and $m = k$ when $r = n$, we have, by $T_A$ (i), (ii) and (iii),

$$y^r \psi^{(r-n)}(y)\{\phi(y)\}^{-\mu} \prod_{\nu=1}^{n} \{\phi^{(\nu)}(y)\}^{\alpha_\nu} = O\left(y^r \frac{\{\gamma(y)\}^{m-r+n}}{y^m} \{\phi(y)\}^{-\mu} \frac{\{\phi(y)\}^{\mu}}{\{\gamma(y)\}^{n}}\right)$$

$$= O\left(\left\{\frac{\gamma(y)}{y}\right\}^{m-r}\right)$$

$$= O(1),$$

$m-r$ being positive.

Hence $I_1$ is of bounded variation with respect to $w$ in the range $[a, \infty)$.

Consider now $I_2$. In view of (1) and Lemmas 3 and 4 (i), in order to prove that $I_2$ is of bounded variation with respect to $w$ in the range $[a, \infty)$, it is sufficient to prove that

$$t^k \int_t^w (u-t)^{p-k} \left(\frac{\partial}{\partial u}\right)^{p+1} \left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^k \psi(u)\right) du \ \ddagger$$

is of uniformly bounded variation with respect to $w$ in the range $[t, \infty)$ for $t \geqslant a$.

Now, in view of Lemma 1 and Leibnitz's theorem on the differentiation of a product, it is sufficient to prove that each of the following integrals is of uniformly bounded variation with respect to $w$ in the range $[t, \infty)$ for $t \geqslant a$:

$$I_{21} = \left\{\frac{t}{\phi(w)}\right\}^k \int_t^w (u-t)^{p-k} \psi^{(p+1)}(u)\{\phi(w)-\phi(u)\}^k du,$$

$$I_{22} = \left\{\frac{t}{\phi(w)}\right\}^k \int_t^w (u-t)^{p-k} \psi^{(p+1-r)}(u)\{\phi(w)-\phi(u)\}^{k-\mu} \prod_{\nu=1}^{r} \{\phi^{(\nu)}(u)\}^{\alpha_\nu} du,$$

and

$$I_{23} = \left\{\frac{t}{\phi(w)}\right\}^k \int_t^w (u-t)^{p-k} \psi(u)\{\phi(w)-\phi(u)\}^{k-\sigma} \prod_{\nu=1}^{p+1} \{\phi^{(\nu)}(u)\}^{\beta_\nu} du,$$

† Products and sums, $\prod_{\nu=1}^{0}$, $\sum_{\nu=1}^{0}$, are taken to have values 1, 0, respectively.

‡ Here and subsequently, integrals of this form, are defined to have value zero when $w = t$.



where $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_{p+1}$ are non-negative integers such that

$$1 \leqslant \sum_{\nu=1}^{r} \alpha_\nu = \mu \leqslant \sum_{\nu=1}^{r} \nu\alpha_\nu = r \leqslant p,$$

and

$$1 \leqslant \sum_{\nu=1}^{p+1} \beta_\nu = \sigma \leqslant \sum_{\nu=1}^{p+1} \nu\beta_\nu = p+1.$$

Now, for $t \geqslant a$, by $T_A$ (i) and (ii)

$$\int_t^\infty t^k(u-t)^{p-k}\psi^{(p+1)}(u)\, du = O\left\{\int_t^\infty t^k(u-t)^{p-k}u^{-p-1}\, du\right\}.$$

and the latter integral is finite and independent of $t$. Hence, in view of Lemma 4 (i), it is clear that $I_{21}$ is of uniformly bounded variation with respect to $w$ in the range $[t, \infty)$ for $t \geqslant a$.

Further, for $w \geqslant t \geqslant a$, we can write

$$I_{22} = \int_t^w \Phi_\mu(w, t)\, t^k(u-t)^{p-k}\psi^{(p+1-r)}(u)\{\phi(u)\}^{-\mu} \prod_{\nu=1}^{r} \{\phi^{(\nu)}(u)\}^{\alpha_\nu} du.$$

By $T_A$ (ii) and (iii), for $t \geqslant a$,

$$\int_t^\infty t^k(u-t)^{p-k}\psi^{(p+1-r)}(u)\{\phi(u)\}^{-\mu} \prod_{\nu=1}^{r} \{\phi^{(\nu)}(u)\}^{\alpha_\nu} du$$

$$= O\left\{\int_t^\infty t^k(u-t)^{p-k}u^{-p-1}\, du\right\},$$

and the latter integral is finite and independent of $t$. Hence, in view of Lemmas 4 (i) and 5, it is clear that $I_{22}$ is of uniformly bounded variation with respect to $w$ in the range $[t, \infty)$ for $t \geqslant a$.

Finally, for $w \geqslant t \geqslant a$, we can write

$$I_{23} = \left\{\frac{t}{\phi(w)}\right\}^k \int_t^w (u-t)^{p-k}\psi(u)\{\phi(w)-\phi(u)\}^{k-p-1}.$$

$$\times \{\phi(w)-\phi(u)\}^{p+1-\sigma} \prod_{\nu=1}^{p+1} \{\phi^{(\nu)}(u)\}^{\beta_\nu} du.$$

Since $p+1-\sigma$ is a positive integer (or zero), we can expand $\{\phi(w)-\phi(u)\}^{p+1-\sigma}$ by the binomial theorem, giving, for $w \geqslant t \geqslant a$,

$$I_{23} = \sum_{r=0}^{p+1-\sigma} c_r t^k \{\phi(w)\}^{p+1-\sigma-k-r}$$

$$\times \int_t^w (u-t)^{p-k}\{\phi(w)-\phi(u)\}^{k-p-1}\{\phi(u)\}^r \psi(u) \prod_{\nu=1}^{p+1} \{\phi^{(\nu)}(u)\}^{\beta_\nu} du,$$

where $c_0, c_1, \ldots, c_{p+1-\sigma}$ are constants.

To the typical integral of the above sum, apply the transformation

$$u = t+vx;\ w = t+x$$

to obtain a constant multiple of $q_4(x, t)$, where, for $t \geqslant a$, $q_4(0, t) = 0$ and for $x > 0$ and $t \geqslant a$

$$q_4(x, t) = t^k\{\phi(x+t)\}^{p+1-\sigma-k-r} \int_0^1 \left\{\frac{\phi(t+x)-\phi(t+vx)}{x(1-v)}\right\}^{k-p-1} \{\phi(t+vx)\}^r$$

$$\times \psi(t+vx) \prod_{\nu=1}^{p+1} \{\phi^{(\nu)}(t+vx)\}^{\beta_\nu}\, v^{p-k}(1-v)^{k-p-1}\, dv.$$

It is sufficient to prove that $q_4(x, t)$ is of uniformly bounded variation with respect to $x$ in the range $[0, \infty)$ for $t \geqslant a$.

Now, for $x > 0$ and $t \geqslant a$, we can write

$$q_4(x, t) = \int_0^1 v^{p-k}(1-v)^{k-p-1}\left\{\frac{t}{t+vx}\right\}^k H_1(x, t, v)$$

$$\times \{H_2(x, t, v)\}^{k-p-1+\sigma+r} H_3(t+vx) \prod_{\nu=1}^{p+1} \{Q_\nu(t+vx)\}^{\beta_\nu}\, dv$$

where

$$H_1(x, t, v) = \left\{\frac{\phi(t+x)-\phi(t+vx)}{x(1-v)\,\phi'(t+vx)}\right\}^{k-p-1},$$

$$H_2(x, t, v) = \frac{\phi(t+vx)}{\phi(t+x)},$$

$$H_3(y) = \left\{\frac{y\phi'(y)}{\phi(y)}\right\}^k \psi(y)$$

and

$$Q_\nu(y) = \left\{\frac{\phi(y)}{\phi'(y)}\right\}^{\nu-1} \frac{\phi^{(\nu)}(y)}{\phi'(y)}.$$

Now

$$\int_0^1 v^{p-k}(1-v)^{k-p-1}\, dv = \Gamma(p+1-k)\,\Gamma(k-p),$$

and each of the remaining terms in the integrand of $q_4(x, t)$ is of uniformly bounded variation with respect to $x$ in the range $[\epsilon, \infty)$ for $\epsilon > 0$, $0 < v < 1$ and $t \geqslant a$;

$\left\{\frac{t}{t+vx}\right\}^k$ being trivially so;

$H_1(x, t, v)$, by $T_B$ (v);

$\{H_2(x, t, v)\}^{k-p-1+\sigma+r}$, by $T_B$ (v) since $\sigma + r \geqslant 1$;

$H_3(t+vx)$, by $T_A$ (iv); and

$Q_\nu(t+vx)$ by $T_A$ (iii)(b).†

† We use the result, that if $\alpha > 1$ and if $f(x)$ is of bounded variation in the range $[\epsilon, \infty)$, then so is $\{f(x)\}^\alpha$



Hence, in view of Lemma 4 (iii), $q_4(x, t)$ is of uniformly bounded variation with respect to $x$ in the range $[\epsilon, \infty)$ for $\epsilon > 0$ and $t \geqslant a$. Further, for $t \geqslant a$, in view of $T_A$ (ii) and (iii),

$$\lim_{x\to 0+} q_4(x, t) = \Gamma(p+1-k)\,\Gamma(k-p)\, H_3(t) \prod_{\nu=1}^{p+1} \{Q_\nu(t)\}^{\beta_\nu}$$

$$= O(1).$$

Consequently, $q_4(x, t)$ is of uniformly bounded variation with respect to $x$ in the range $[0, \infty)$ for $t \geqslant a$, and so, $I_{23}$ is of uniformly bounded variation with respect to $w$ in the range $[t, \infty)$ for $t \geqslant a$.

Hence $I_2$ is of bounded variation with respect to $w$ in the range $[a, \infty)$, and it follows that (2) is of bounded variation for $w \geqslant a$, this completing the proof of $T_B$.

## 4. *Proof of* $T_A$.

In view of Lemma 6, $T_A$ follows immediately from $T_B$.

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