ON MULTIPLICATION OF CESARO SUMMABLE SERIES

D. Borwein *and* Y. Matsuoka

1. *Introduction*

Throughout this paper $\sum_0^\infty c_n$ denotes the Cauchy product of the series $\sum_0^\infty a_n$ and $\sum_0^\infty b_n$, *i.e.*

$$c_n = \sum_{r=0}^{n} a_r b_{n-r};$$

and (C, α), $[C, \alpha]$, $|C, \alpha|$ denote respectively ordinary, strong and absolute Cesàro summability methods. The method $[C, \alpha]$, previously defined only for $\alpha \geqslant 0$, is defined in a natural way for $\alpha < 0$ in §2.

It is known (see [1] and the references there given) that *if* $\sum_0^\infty a_n$ *is summable* $|C, -\mu|$ *to* A *and* $\sum_0^\infty b_n$ *is summable* $(C, -\mu)$ *to* B, *where* $\mu \geqslant 0$, *then* $\sum_0^\infty c_n$ *is summable* $(C, -\mu)$ *to* AB.

As a companion to this result we prove:

Theorem 1. *If* $\mu \geqslant 0$ *and* $\sum_0^\infty a_n$, $\sum_0^\infty b_n$ *are summable* $[C, -\mu]$ *to* A, B *respectively, then* $\sum_0^\infty c_n$ *is summable* $(C, -\mu)$ *to* AB.

The case $\mu = 0$ of this theorem has been established by Boyd [4]. We also prove the following two theorems.

Theorem 2. *There are series* $\sum_0^\infty a_n$, $\sum_0^\infty b_n$, *respectively summable* $(C, -1)$ *and absolutely convergent, for which* $\sum_0^\infty c_n$ *is not summable* $[C, 0]$.

Theorem 3. *Given* $\alpha \geqslant -1$, *there are series* $\sum_0^\infty a_n$, $\sum_0^\infty b_n$, *respectively summable* $(C, -1)$ *and* (C, α), *for which* $\sum_0^\infty c_n$ *is not summable* $[C, \alpha+1]$.

The cases $\alpha = -1$ and $\alpha = 0$ of Theorem 3 have been proved by Boyd [4]. Immediate consequences of Theorems 2 and 3 respectively (see inclusion IV in §2) are:

Corollary 1. *There are series* $\sum_0^\infty a_n$, $\sum_0^\infty b_n$, *respectively summable* $[C, 0]$ *and absolutely convergent, for which* $\sum_0^\infty c_n$ *is not summable* $[C, 0]$.

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Corollary 2. *Given $l \geqslant 0$, there are series $\sum_0^\infty a_n$, $\sum_0^\infty b_n$, respectively summable* [C, 0] *and* [C, l], *for which $\sum_0^\infty c_n$ is not summable* [C, l].

We state next three known propositions, the first due to Boyd [4] and the others to Winn [7].

(α) *If $\sum_0^\infty a_n$ is summable* [C, k] *to A, where $k > 0$, and $\sum_0^\infty b_n$ is absolutely convergent with sum B, then $\sum_0^\infty c_n$ is summable* [C, k] *to AB.*

(β) *If $\sum_0^\infty a_n$ is summable* [C, k] *to A and $\sum_0^\infty b_n$ is summable* (C, l) *to B, where $k > 0$, $l \geqslant 0$, then $\sum_0^\infty c_n$ is summable* (C, $k+l$) *to AB.*

(γ) *If $\sum_0^\infty a_n$ is summable* [C, k] *to A and $\sum_0^\infty b_n$ is summable* [C, l] *to B, where $k > 0$, $l > 0$, then $\sum_0^\infty c_n$ is summable* [C, $k+l$] *to AB.*

Corollary 1 shows that proposition (α) fails when $k > 0$ is replaced by $k = 0$. Boyd [4] has demonstrated the falsity of (β) with $k = l = 0$ in place of $k > 0, l \geqslant 0$, and of (γ) with $k = 0$, $l = 0$ or 1 in place of $k > 0, l > 0$. Corollary 2 shows that for every $l_0 \geqslant 0$, (γ) is false when $k > 0$, $l > 0$ is replaced by $k = 0$, $l = l_0$.

2. *Notation, definitions and preliminary results.*

Let
$$s_n = \sum_{r=0}^n a_r \quad (n = 0, 1, \ldots).$$

Given matrices $Q = (q_{n,r})$, $P = (p_{n,r})$ $(n, r = 0, 1, \ldots)$ with $p_{n,r} \geqslant 0$, the strong summability method [P, Q] is defined (see [3]) as follows. Let
$$\sigma_n = Q(s_n) = \sum_{r=0}^\infty q_{n,r}\, s_r.$$
Then $\sum_0^\infty a_n$ is summable [P, Q] to s, and we write $s_n \to s[P, Q]$, if
$$\sum_{r=0}^\infty p_{n,r} |\sigma_r - s|$$
is defined for each n and tends to 0 as $n \to \infty$.

We use the notation:
$$\epsilon_n^\alpha = \binom{n+\alpha}{n}, \quad \Delta^\alpha s_n = \sum_{r=0}^n \epsilon_{n-r}^{-\alpha-1} s_r \quad (n = 0, 1, \ldots; \text{ any real } \alpha).$$
Denote by $C_{\alpha,\beta}$ the matrix of the linear transformation from $\{s_n\}$ to $\{\sigma_n\}$ given by
$$\sigma_n = \frac{1}{\epsilon_n^{\alpha+\beta}} \sum_{r=0}^n \epsilon_{n-r}^{\alpha-1} \epsilon_r^\beta s_r = \frac{1}{\epsilon_n^{\alpha+\beta}} \Delta^{-\alpha}(\epsilon_n^\beta s_n) \quad (\beta > -1, \alpha+\beta > -1);$$

it is known ([2], Theorem 8) that $C_{\alpha,\beta}$ is the Hausdorff matrix generated by the sequence $\{\epsilon_n^\beta / \epsilon_n^{\alpha+\beta}\}$.

Define C_α to be the matrix $C_{\alpha,0}$ when $\alpha > -1$, and $C_{\alpha,-\alpha}$ when $\alpha \leqslant -1$. Then, for any real α, the statement
$$\sum_0^\infty a_n \text{ is summable } (C, \alpha) \text{ to } A$$
can be interpreted (see [1], 443) as
$$\sigma_n = C_\alpha(s_n) \to A.$$

We now define, for every real α, the strong Cesàro method [C, α] to be [C_1, $C_{\alpha-1}$]. The definition is standard for $\alpha > 0$; for $\alpha < 0$, the method [C, α] does not appear to have been defined explicitly before. The following proposition, which is a special case of a known result ([3], III) with $X = C_{-1,1}$), shows that our definition of [C, 0] is equivalent to one framed by Hyslop [6].

I. *The series $\sum_0^\infty a_n$ is summable* [C, 0] *to A if and only if it is convergent with sum A and*
$$\sum_{r=0}^n r|a_r| = o(n).$$

Given summability methods X, Y we say that X is included in Y and write $X \subseteq Y$ if every series summable X is also summable Y to the same sum; X and Y are said to be equivalent and we write $X \simeq Y$ if each is included in the other.

We list next some inclusions, which hold for every real α, together with references to results of which they are immediate consequences.

II. $[C, \alpha] \simeq [C_1, C_{\alpha-1,\beta}]$ $(\beta > -1, \alpha+\beta > 0)$.
([3], II; and [2], Theorem 9.)

III. $[C, \alpha] \subseteq [C, \alpha+\delta]$ $(\delta > 0)$.
([3], II; and [2], Theorem 6.)

IV. $(C, \alpha-1) \subseteq [C, \alpha] \subseteq (C, \alpha)$.
([3], Theorem 3; and [2], Theorem 7.)

V. $|C, \alpha| \subseteq [C, \alpha]$.
([3], Theorem 12; and [1], |III|.)

That III, IV and V hold for $\alpha \geqslant 0$ was known before (see [4], [6], [7]). Inclusion V is listed for interest only and is not used in the rest of this paper.

3. *Proofs of the theorems.*

In order to prove Theorem 1 we require a lemma which is similar to one proved by Winn ([7], 483–484).

LEMMA. *If* $W_n = \sum_{r=0}^{n} w_r = o(n)$ *then, for* $\alpha < 1$, $\sum_{r=0}^{n} \epsilon_r^{-\alpha} w_r = o(\epsilon_n^{1-\alpha})$.

Proof. By partial summation we have

$$\sum_{r=0}^{n} \epsilon_r^{-\alpha} w_r = \sum_{r=0}^{n} W_r(\epsilon_r^{-\alpha} - \epsilon_{r+1}^{-\alpha}) + W_n \epsilon_{n+1}^{-\alpha} = \alpha \sum_{r=0}^{n} \epsilon_r^{-\alpha} W_r/(r+1) + o(\epsilon_n^{1-\alpha}).$$

Since $W_r/(r+1) \to 0$ and $\sum_{0}^{n} \epsilon_r^{-\alpha} = \epsilon_n^{1-\alpha}$, the required result can now be obtained by an application of Toeplitz's theorem.

Proof of Theorem 1.

Case (i). Suppose $A = B = 0$.

Let $\mu = m + \alpha$, where m is a non-negative integer and $0 \leqslant \alpha < 1$; and let

$$s_n = \sum_{r=0}^{n} a_r, \quad t_n = \sum_{r=0}^{n} b_r.$$

It has been shown ([1], 447) that *a necessary and sufficient condition for* $\sum_0^\infty c_n$ *to be summable* $(C, -\mu)$ *to* 0 *is that*

$$X_n + Y_n + Z_n = o(1),$$

where

$$X_n = \sum_{\rho=1}^{m} \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \Delta^{m+1-\rho}(\epsilon_r^{m+1-\rho} s_r) \ \Delta^{\rho+\alpha}(\epsilon_{n-r}^{\rho} t_{n-r})$$

when $m \geqslant 1$ *and* $X_n = 0$ *when* $m = 0$, *and*

$$Y_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} t_{n-r} \Delta^{\mu+1}(\epsilon_r^{\mu+1-\alpha} s_r),$$

$$Z_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} s_{n-r} \Delta^{\mu+1}(\epsilon_r^{\mu+1-\alpha} t_r).$$

By hypothesis, $s_n \to 0$ $[C, -\mu]$, $t_n \to 0$ $[C, -\mu]$, so that by the second inclusion in IV (§2),

$$s_n \to 0 \ (C, -\mu), \quad t_n \to 0 \ (C, -\mu);$$

and a known consequence ([1], 447–448) is that

$$X_n = o(1).$$

Now let

$$y_n = \Delta^{\mu+1}(\epsilon_n^{\mu+1-\alpha} s_n) = \epsilon_n^{-\alpha} C_{-\mu-1, \mu+1-\alpha}(s_n).$$

From the hypothesis $s_n \to 0$ $[C, -\mu]$ we deduce, by II, that

$$\sum_{r=0}^{n} \frac{|y_r|}{\epsilon_r^{-\alpha}} = o(n)$$

and hence, by the Lemma, that

$$\frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} |y_r| = o(1).$$

Next, since $t_n \to 0$ $[C, -\mu]$ we have, by III, that $t_n = o(1)$ and it follows that

$$Y_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} t_{n-r} y_r = o(1).$$

Similarly $Z_n = o(1)$; and the proof of Case (i) is complete.

Case (ii). Suppose now that there are no restrictions on A, B. Let $a_0' = a_0 - A$, $b_0' = b_0 - B$; $a_r' = a_r$, $b_r' = b_r$ $(r > 0)$ and let

$$c_n' = \sum_{r=0}^{n} a_r' b_{n-r}'.$$

Since $\sum_0^\infty a_n$, $\sum_0^\infty b_n$ are summable $[C, -\mu]$ to A, B respectively, it is readily seen that $\sum_0^\infty a_n'$ and $\sum_0^\infty b_n'$ are both summable $[C, -\mu]$ to 0, from which it follows, by Case (i), that

$$\sum_0^\infty c_n' \text{ is summable } (C, -\mu) \text{ to } 0.$$

But

$$\sum_0^N c_n = \sum_0^N c_n' + B \sum_0^N a_n + A \sum_0^N b_n - AB,$$

and $\sum_0^\infty a_n$, $\sum_0^\infty b_n$ are summable $(C, -\mu)$ to A, B respectively. Hence $\sum_0^\infty c_n$ is summable $(C, -\mu)$ to AB. This completes the proof.

Proof of Theorem 2. For convenience we divide the proof into three parts.

Part (i). Let $u_n \geqslant 0$, $v_n \geqslant 0$, $U_n = \sum_0^n u_r$, $V_n = \sum_0^n v_r$ $(n = 0, 1, \ldots)$, and let $a_n = (-1)^n u_n$, $b_n = (-1)^n v_n$. Then

$$c_\nu = \sum_{r=0}^{\nu} a_r b_{\nu-r} = (-1)^\nu \sum_{r=0}^{\nu} u_r v_{\nu-r},$$

and hence ([4], 30)

$$\sum_{\nu=0}^{2n} \nu |c_\nu| = \sum_{\nu=0}^{2n} \nu \sum_{r=0}^{\nu} u_r v_{\nu-r} = \sum_{r=0}^{2n} r u_r V_{2n-r} + \sum_{r=0}^{2n} r v_r U_{2n-r}$$

$$\geqslant \sum_{r=0}^{n} r v_r U_{2n-r} \geqslant U_n \sum_{r=0}^{n} r v_r. \tag{1}$$

Part (ii). We show now that given any unbounded sequence of positive numbers $\{U_n\}$, there is a sequence $\{v_n\}$ such that

$$v_n \geqslant 0,\ \sum_0^\infty v_n < \infty \text{ and } U_n \sum_0^n r v_r \neq o(n). \tag{2}$$

Let $\{\beta_n\}$ be a sequence not converging to 0 such that

$$\beta_n \geqslant 0 \text{ and } \sum_0^\infty \frac{\beta_n}{U_n} < \infty;$$

a suitable sequence can be constructed by first defining an increasing sequence of positive integers $\{n_\nu\}$ for which

$$U_{n_\nu} > \nu^2,$$

and then taking β_n to be 1 whenever $n = n_\nu$ and 0 otherwise.

Let

$$\alpha_0 = 0,\ \alpha_n = \frac{\beta_n}{U_n} - \frac{n-1}{n}\frac{\beta_{n-1}}{U_{n-1}} \quad (n \geqslant 1).$$

Then

$$U_n \sum_0^n r\alpha_r = n\beta_n$$

and

$$\sum_0^\infty |\alpha_n| < \infty.$$

Setting $v_n = |\alpha_n|$, we have

$$U_n \sum_0^n r v_r \geqslant n\beta_n,$$

and so the sequence $\{v_n\}$ satisfies (2) as required.

Part (iii). To prove our theorem take $a_n = (-1)^n u_n$ where $u_n > 0$, $nu_n = o(1)$ and $\sum_0^\infty (-1)^n u_n$ is conditionally convergent; *e.g.* $u_n = 1/\big((n+2)\log(n+2)\big)$. Then $U_n = \sum_0^n u_r$ is positive and tends to infinity, and $\sum_0^\infty a_n$ is summable $(C, -1)$. Let $b_n = (-1)^n v_n$ where $\{v_n\}$ is a sequence satisfying (2); then $\sum_0^\infty b_n$ is absolutely convergent. In virtue of I, the Cauchy product $\sum_0^\infty c_n$ of the above series $\sum_0^\infty a_n$, $\sum_0^\infty b_n$ is not summable $[C, 0]$, since, by (1) and (2),

$$\sum_{\nu=0}^{2n} \nu|c_\nu| \neq o(n).$$

Proof of Theorem 3. Since the case $\alpha = -1$ has been proved by Boyd [4] we may suppose that $\alpha > -1$. Let

$$f(x) = \frac{x^{\alpha+1}}{\log\log x};$$

then, as $x \to \infty$,

$$f'(x) = \frac{(\alpha+1)x^\alpha}{\log\log x}\big(1+o(1)\big) \tag{3}$$

and so there is a positive integer p such that

$$f(x+1) > f(x) > 0 \text{ for } x \geqslant p.$$

Let

$$\delta_{-1} = \delta_0 = \ldots = \delta_{p-1} = 0,\quad \delta_n = f(n) \quad (n \geqslant p)$$

and let

$$\beta_n = \frac{(-1)^n}{\epsilon_n^{\alpha}}(\delta_n - \delta_{n-1}) \quad (n \geqslant 0).$$

Then, for $n > p$,

$$\beta_n = \frac{(-1)^n}{\epsilon_n^{\alpha}} f'(n-1+\theta) \quad (0 < \theta < 1),$$

and so, by (3),

$$\beta_n = O\left(\frac{1}{\log\log n}\right) = o(1) \text{ as } n \to \infty.$$

Now set

$$a_0 = a_1 = 0,\quad a_n = \frac{(-1)^n}{n\log n} \quad (n \geqslant 2),$$

$$B_n = \sum_{r=0}^n b_r = \Delta^\alpha(\epsilon_n^{\alpha}\beta_n).$$

Then $\sum_0^\infty a_n$ is summable $(C, -1)$; and, since $C_\alpha(B_n) = \beta_n \to 0$, $\sum_0^\infty b_n$ is summable (C, α) to 0. Let

$$c_n = \sum_{r=0}^n a_r b_{n-r},\quad \gamma_n = \sum_{r=0}^n c_r,\quad \sigma_n = C_\alpha(\gamma_n).$$

Then

$$\gamma_n = \sum_{r=0}^n a_{n-r} B_r = \sum_{r=0}^n a_{n-r}\Delta^\alpha(\epsilon_r^{\alpha}\beta_r),$$

$$\epsilon_n^{\alpha}\sigma_n = \sum_{r=0}^n a_{n-r}\epsilon_r^{\alpha}\beta_r = (-1)^n \sum_{r=0}^n |a_{n-r}|(\delta_r - \delta_{r-1});$$

and so, as $m \to \infty$,

$$\sum_{n=0}^{2m} \epsilon_n^{\alpha}|\sigma_n| = \sum_{n=0}^{2m} |a_n|\delta_{2m-n} > \sum_{n=0}^{m} |a_n|\delta_{2m-n} > \frac{m^{\alpha+1}}{\log\log m}\sum_{n=2}^m \frac{1}{n\log n} \sim m^{\alpha+1}.$$

It follows that

$$\sum_{n=0}^{m} \epsilon_n^{\alpha} |\sigma_n| \neq o(m^{\alpha+1})$$

and hence, by our Lemma, that

$$\sum_{n=0}^{m} |\sigma_n| \neq o(m).$$

Consequently $\sum_0^\infty c_n$ is not summable $[C, \alpha+1]$ to 0. However, by a standard result ([5], Theorem 164), $\sum_0^\infty c_n$ is summable $(C, \alpha+1)$ to 0 and so, by the second inclusion in IV (§2), the series cannot be summable $[C, \alpha+1]$ to any number other than 0. Hence $\sum_0^\infty c_n$ is not summable $[C, \alpha+1]$.

Remark. It is known ([5], Theorem 166) that, *given* $\alpha \geqslant -1$, *there are series* $\sum_0^\infty a_n$, $\sum_0^\infty b_n$, *respectively summable* $(C, -1)$ *and* (C, α), *for which* $\sum_0^\infty c_n$ *is not summable* (C, α).

Our Theorem 3 is stronger than this result, since (C, α) is included in, but is not equivalent to, $[C, \alpha+1]$.

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St. Salvator's College,
University of St. Andrews.

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