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ON RIESZ SUMMABILITY FACTORS

by D. BORWEIN and B. L. R. SHAWYER

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1. Suppose throughout that a, k are positive numbers and that p is the integer such that $k-1 \leqq p < k$. Suppose also that $\phi(w)$, $\psi(w)$ are functions with absolutely continuous $(p+1)$th derivatives in every interval $[a, W]$ and that $\phi(w)$ is positive and unboundedly increasing. Let $\lambda = \{\lambda_n\}$ be an unboundedly increasing sequence with $\lambda_1 > 0$.

Given a series $\sum_{n=1}^{\infty} a_n$, and a number $m \geqq 0$, we write

$$A_m(w) = \begin{cases} \sum_{\lambda_n \leqq w} (w-\lambda_n)^m a_n & \text{if } w > \lambda_1, \\ 0 & \text{otherwise,} \end{cases}$$

and $A(w) = A_0(w)$.

If $w^{-m}A_m(w)$ tends to a finite limit as $w \to \infty$, $\sum_{n=1}^{\infty} a_n$ is said to be summable (R, λ, m).

The object of this note is to obtain conditions sufficient to ensure, when k is not an integer, the truth of the proposition

P. $\sum_{n=1}^{\infty} a_n\psi(\lambda_n)$ *is summable* $(R, \phi(\lambda), k)$ *whenever* $\sum_{n=1}^{\infty} a_n$ *is summable* (R, λ, k).

For integral values of k, the following theorem is known [**1**].

T_1. *If*

(i) $\gamma(w)$ *is positive and absolutely continuous in every interval* $[a, W]$ *and* $\gamma'(w) = O(1)$ *for* $w \geqq a$,

(ii) $w^n\psi^{(n)}(w) = O\left\{\left(\frac{\gamma(w)}{w}\right)^{k-n}\right\}$ $(n = 0, 1, \ldots, k;\ w \geqq a)$,

(iii) $\int_a^{\infty} t^k \mid \psi^{(k+1)}(t) \mid dt < \infty$,

(iv) $\int_a^{w} \{\gamma(t)\}^n \mid \phi^{(n+1)}(t) \mid dt = O\{\phi(w)\}$ $(n = 1, 2, \ldots, k;\ w \geqq a)$,

then P.

Other known theorems, which hold for all $k \geqq 0$, are

T_2. *If* $\phi(w) = e^w$ *and* $\psi(w) = w^{-k}$, *then* P;

T_3. *If*

(i) $\phi(w)$ *is a logarithmico-exponential function,*

(ii) $\frac{1}{w} \prec \frac{\phi'(w)}{\phi(w)} \prec 1$,

(iii) $\psi(w) = \left\{\frac{\phi(w)}{w\phi'(w)}\right\}^k$,

then P;

and T_3', which is more general than T_3, in that hypothesis (ii) is replaced by

(ii)' $\frac{1}{w} \preccurlyeq \frac{\phi'(w)}{\phi(w)}$.

T_2, which is included in T_3', is a well known theorem of Hardy [**4**, 30] and T_3 and T_3' are due to Guha [**2**], who derived the latter from the former by means of standard results. For integral values of k, the hypotheses of T_1 are satisfied when $\phi(w)$, $\psi(w)$ are as in T_3' and $\gamma(w) = \phi(w)/\phi'(w)$.

Suppose, from now on, that k is not an integer. We shall prove the following theorems as companions to T_1.

T_A. *If*

(i) $\gamma(w)$ *is positive and absolutely continuous in every interval* $[a, W]$, *and* $\gamma'(w) = O(1)$ *for* $w \geqq a$,

(ii) (*a*) $\psi(w) = O\left(\left\{\frac{\gamma(w)}{w}\right\}^k\right)$ *for* $w \geqq a$,

(*b*) $w^n\psi^{(n)}(w) = O\left(\left\{\frac{\gamma(w)}{w}\right\}^{p+1-n}\right)$ *for* $n = 1, 2, \ldots, p+1$ *and* $w \geqq a$,

(iii) $\int_a^{\infty} t^{p+1} \mid \psi^{(p+2)}(t) \mid dt < \infty$,

(iv) $\phi'(w)$ *is positive monotonic non-decreasing for* $w \geqq a$,

(v) $\gamma(w)\phi'(w) = O\{\phi(w)\}$ *for* $w \geqq a$ *or* $\{\gamma(w)\}^{n-1}\phi^{(n)}(w)/\phi'(w)$ *is of bounded variation in* $[a, \infty)$ *for* $n = 1, 2, \ldots, p+1$ *according as* $0 < k < 1$ *or* $k > 1$,

(vi) $\phi''(w)/\phi'(w)$ *is monotonic non-increasing for* $w \geqq a$,

(vii) $h_n(w) = \psi(w)\{\phi'(w)\}^{k-n}\{\gamma(w)\}^{-n}$ *is positive monotonic in the range* $w \geqq a$ *for* $n = 0, 1, \ldots, p$, *possibly in different senses for different values of* n,

(viii) $\phi(w) > c\, w^{k/(k-p)}$ *for* $w \geqq a$, *where* c *is a positive constant,*

then P.

T_B. *If* T_A (i) *to* T_A (vii) *inclusive hold, and, in addition,*

(vii)' $h_p(w)$ *is non-decreasing,*

then P.

It is evident that T_2, for non-integral k, is included in T_A, and it can readily be shown that, under the hypotheses of T_3, the hypotheses of T_A are satisfied with $\gamma(w) = \phi(w)/\phi'(w)$ and $\phi(w)$, $\psi(w)$ as in T_3.

We are indebted to the referee for valuable suggestions which led to the above formulation of the results. In the original version of our manuscript we proved that P is a consequence of conditions T_A (i) to T_A (vi) inclusive together with the condition that $h_n(w)$ is a positive monotonic non-decreasing function of w in the range $w \geqq a$ for $n = 0, 1, \dots, p$. The argument in § 4 is due to the referee: it shows that the conditions of T_B are in fact more stringent than those of T_A.

2. The following lemmas are required.

Lemma 1. *If* T_A(i) *and* T_A(v), *then for* $n = 1, 2, \dots, p+1$ *and* $w \geqq a$,

$$\int_a^w \{\gamma(t)\}^{n-1} \,|\, \phi^{(n)}(t) \,|\, dt = O\{\phi(w)\} \tag{2.1}$$

and

$$\{\gamma(w)\}^n \phi^{(n)}(w) = O\{\phi(w)\}. \tag{2.2}$$

Proof. When $0 < k < 1$, (2.2) is the same as the operative hypothesis in T_A (v) and (2.1) is a trivial consequence. Suppose that $k > 1$. Then (2.1) follows from the appropriate part of T_A (v) by integration; hence

$$\gamma(w)\phi'(w) = \gamma(a)\phi'(a) + \int_a^w \gamma(t)\phi''(t)\, dt + \int_a^w \gamma'(t)\phi'(t)\, dt = O\{\phi(w)\},$$

since $\gamma'(t) = O(1)$, and (2.2) is an immediate consequence. (Cf. [**1**, Lemma 2].)

Lemma 2. *The nth derivative of* $\{g(t)\}^m$ *is a sum of a number of terms like*

$$A\{g(t)\}^{m-\sigma} \prod_{\nu=1}^{n} \{g^{(\nu)}(t)\}^{\alpha_\nu},$$

where A is a constant, and $\alpha_1, \alpha_2, \dots, \alpha_n$ *are non-negative integers, such that*

$$1 \leqq \sum_{\nu=1}^{n} \alpha_\nu = \sigma \leqq \sum_{\nu=1}^{n} \nu\alpha_\nu = n.$$

This is a particular case of a theorem due to Faa di Bruno [**5**, I, pp. 89–90].

Lemma 3. *If* a_n *is real,* $a \leqq \xi \leqq w$, *then*

$$\frac{\Gamma(k+1)}{\Gamma(p+1)\Gamma(k-p)} \left| \int_a^\xi A_p(t)(w-t)^{k-p-1}\, dt \right| \leqq \max_{a \leqq t \leqq \xi} |\, A_k(t) \,|.$$

A proof of this lemma has been given by Hardy and Riesz [**4**, 28].

Lemma 4. *If*

$$\overline{\lim_{w\to\infty}} \int_a^w |\, f(w,t) \,|\, dt < \infty \quad \textit{and} \quad \lim_{w\to\infty} \int_a^y |\, f(w,t) \,|\, dt = 0$$

for every finite $y > a$, *and if* $s(t)$ *is a bounded measurable function in* (a, ∞) *which tends to zero as t tends to infinity, then*

$$\lim_{w\to\infty} \int_a^\infty f(w,t)s(t)\, dt = 0.$$

For a proof of this simple result see [**3**, 50] or [**1**, Lemma 3].

Lemma 5. *If* T_A(iv) *and* T_A(vi), *then*

$$\chi(t) = \frac{1}{\phi'(t)} \cdot \frac{\phi(w)-\phi(t)}{w-t}$$

is a monotonic non-increasing function of t for $a \leqq t < w$.

Proof. We have, for $a \leqq t < w$,

$$\begin{aligned}
\frac{\chi'(t)}{\chi(t)} &= \frac{\{\phi(w)-\phi(t)\} - (w-t)\phi'(t)}{\{\phi(w)-\phi(t)\}(w-t)} - \frac{\phi''(t)}{\phi'(t)} \\
&= \frac{\phi'(\eta)-\phi'(t)}{\phi(w)-\phi(t)} - \frac{\phi''(t)}{\phi'(t)} \qquad (w > \eta > t) \\
&\leqq \frac{\phi'(w)-\phi'(t)}{\phi(w)-\phi(t)} - \frac{\phi''(t)}{\phi'(t)} \\
&= \frac{\phi''(\xi)}{\phi'(\xi)} - \frac{\phi''(t)}{\phi'(t)} \qquad (w > \xi > t) \\
&\leqq 0.
\end{aligned}$$

Since $\chi(t) \geqq 0$, the result follows.

3. Proof of T_A. We assume, without loss of generality, that

$$A(w) = 0 \quad \text{for} \quad 0 \leqq w \leqq a$$

and

$$A_k(w) = o(w^k), \tag{3.1}$$

and note that, for $w \geqq a$, it is sufficient to prove that

$$\sum_{\phi(\lambda_n) \leqq \phi(w)} \left\{1 - \frac{\phi(\lambda_n)}{\phi(w)}\right\}^k \psi(\lambda_n) a_n,$$

which is equal to

$$\int_a^w \left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\, dA(t), \tag{3.2}$$

tends to a finite limit as $w \to \infty$. After $p+1$ integrations by parts, (3.2) reduces to a constant multiple of

$$\int_a^w A_p(t)\left(\frac{\partial}{\partial t}\right)^{p+1}\left(\left\{1-\frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right)dt$$

which, by Lemma 2 and Leibnitz's theorem on the differentiation of a product, can be expressed as a sum of constant multiples of integrals of the types

$$I_1 = \{\phi(w)\}^{-k}\int_a^w A_p(t)\psi^{(p+1)}(t)\,\{\phi(w)-\phi(t)\}^k\,dt,$$

$$I_2 = \{\phi(w)\}^{-k}\int_a^w A_p(t)\psi^{(p+1-r)}(t)\,\{\phi(w)-\phi(t)\}^{k-\sigma}\prod_{\nu=1}^{r}\{\phi^{(\nu)}(t)\}^{\alpha_\nu}\,dt$$

and

$$I_3 = \{\phi(w)\}^{-k}\int_a^w A_p(t)\psi(t)\,\{\phi(w)-\phi(t)\}^{k-\rho}\prod_{\nu=1}^{p+1}\{\phi^{(\nu)}(t)\}^{\beta_\nu}\,dt,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_{p+1}$ are non-negative integers such that

$$1 \leqq \sum_{\nu=1}^{r}\alpha_\nu = \sigma \leqq \sum_{\nu=1}^{r}\nu\alpha_\nu = r \leqq p,$$

$$1 \leqq \sum_{\nu=1}^{p+1}\beta_\nu = \rho \leqq \sum_{\nu=1}^{p+1}\nu\beta_\nu = p+1.$$

Consider first I_1. Integrate it by parts to obtain

$$I_1 = -I_{11} + kI_{12},$$

where

$$I_{11} = \{\phi(w)\}^{-k}\int_a^w A_{p+1}(t)\psi^{(p+2)}(t)\,\{\phi(w)-\phi(t)\}^k\,dt$$

and

$$I_{12} = \{\phi(w)\}^{-k}\int_a^w A_{p+1}(t)\psi^{(p+1)}(t)\phi'(t)\,\{\phi(w)-\phi(t)\}^{k-1}\,dt.$$

Now, by a standard result [4, 29] and (3.1),

$$A_{p+1}(w) = o(w^{p+1}). \tag{3.3}$$

Hence, using (3.3) and T_A (iii), we obtain

$$\int_a^\infty |\,\psi^{(p+2)}(t)A_{p+1}(t)\,|\,dt < \infty,$$

and so, by Lebesgue's theorem on dominated convergence, I_{11} tends to

$$l = \int_a^\infty \psi^{(p+2)}(t)A_{p+1}(t)\,dt \quad \text{as} \quad w \to \infty,$$

l being finite.

For I_{12}, consider the function

$$f_1(w,t) = \{\phi(w)\}^{-k}t^{p+1}\psi^{(p+1)}(t)\phi'(t)\,\{\phi(w)-\phi(t)\}^{k-1}.$$

Using T_A (ii), we note that, for $w > t \geqq a$,

$$|\,f_1(w,t)\,| < M_1\,\{\phi(w)\}^{-k}\phi'(t)\,\{\phi(w)-\phi(t)\}^{k-1},$$

where M_1 is a constant. Hence $f_1(w,t)$ satisfies the hypotheses of Lemma 4, and so

$$\int_a^w f_1(w,t)t^{-p-1}A_{p+1}(t)\,dt \to 0 \quad \text{as} \quad w \to \infty.$$

That is $\lim_{w\to\infty} I_{12} = 0$ and so

$$\lim_{w\to\infty} I_1 = l. \tag{3.4}$$

Considering now I_2, we see, on integrating by parts, that it is equal to the sum of constant multiples of integrals of the types

$$I_{21} = \{\phi(w)\}^{-k}\int_a^w A_{p+1}(t)\psi^{(p+2-r)}(t)\{\phi(w)-\phi(t)\}^{k-\sigma}\prod_{\nu=1}^{r}\{\phi^{(\nu)}(t)\}^{\alpha_\nu}\,dt,$$

$$I_{22} = \{\phi(w)\}^{-k}\int_a^w A_{p+1}(t)\psi^{(p+1-r)}(t)\,\{\phi(w)-\phi(t)\}^{k-\sigma-1}\phi'(t)\prod_{=1}^{r}\{\phi^{(\nu)}(t)\}^{\alpha_\nu}\,dt$$

and

$$I_{23} = \{\phi(w)\}^{-k}\int_a^w A_{p+1}(t)\psi^{(p+1-r)}(t)\{\phi(w)-\phi(t)\}^{k-\sigma}\prod_{\nu=1}^{r+1}\{\phi^{(\nu)}(t)\}^{\delta_\nu}\,dt$$

where $\alpha_1, \alpha_1, \ldots, \alpha_r, \delta_1, \delta_1, \ldots, \delta_{r+1}$ are non-negative integers, such that

$$1 \leqq \sum_{\nu=1}^{r}\alpha_\nu = \sigma \leqq \sum_{\nu=1}^{r}\nu\alpha_\nu = r \leqq p;$$

$$\sum_{\nu=1}^{r+1}\delta_\nu = \sigma; \quad \sum_{\nu=1}^{r+1}\nu\delta_\nu = r+1.$$

For I_{21}, consider

$$f_2(w,t) = \{\phi(w)\}^{-k}t^{p+1}\psi^{(p+2-r)}(t)\,\{\phi(w)-\phi(t)\}^{k-\sigma}\prod_{\nu=1}^{r}\{\phi^{(\nu)}(t)\}^{\alpha_\nu}.$$

Suppose that the non-vanishing α_ν of highest suffix is α_s. Then

$$f_2(w,t) = \{\phi(w)\}^{-k}t^{p+1}\psi^{(p+2-r)}(t)\phi^{(s)}(t)\,\{\phi(w)-\phi(t)\}^{k-\sigma}\prod_{\nu=1}^{s-1}\{\phi^{(\nu)}(t)\}^{\alpha_\nu}\{\phi^{(s)}(t)\}^{\alpha_s-1}$$

and

$$1 \leqq \sum_{\nu=1}^{s}\alpha_\nu = \sigma \leqq \sum_{\nu=1}^{s}\nu\alpha_\nu = r.$$

Using (2.2) and T_A (ii), we find that, for $w > t \geqq a$,

$$|f_2(w,t)| < M_2\{\phi(w)\}^{-k}t^{p+1}\{\gamma(t)\}^{r-1}t^{-p-1}\,|\phi^{(s)}(t)|\,\{\phi(w)-\phi(t)\}^{k-\sigma}\{\phi(t)\}^{\sigma-1}\{\gamma(t)\}^{s-r}$$
$$< M_2\{\phi(w)\}^{-1}\{\gamma(t)\}^{s-1}\,|\phi^{(s)}(t)|,$$

where M_2 is a constant. Because of (2.1), $f_2(w,t)$ satisfies the hypotheses of Lemma 4, and so

$$\int_a^w f_2(w,t)t^{-p-1}A_{p+1}(t)\,dt \to 0 \quad \text{as} \quad w \to \infty.$$

That is, $\lim_{w\to\infty} I_{21} = 0$. Similarly $\lim_{w\to\infty} I_{23} = 0$, and $\lim_{w\to\infty} I_{22} = 0$ in the case $k-\sigma-1 > 0$. The remaining case of I_{22} is that in which $r = \sigma = p$, and we write the integral as

$$\{\phi(w)\}^{-k}\int_a^w A_{p+1}(t)\psi'(t)\{\phi'(t)\}^{p+1}\{\phi(w)-\phi(t)\}^{k-p-1}\,dt.$$

Consider

$$f_3(w,t) = \{\phi(w)\}^{-k}t^{p+1}\psi'(t)\phi'(t)\{\phi(w)-\phi(t)\}^{k-p-1}\{\phi'(t)\}^p.$$

Using (2.2) and T_A (ii), we find that, for $w > t \geqq a$,

$$|f_3(w,t)| < M_3\{\phi(w)\}^{-k}t^{p+1}\{\gamma(t)\}^p t^{-p-1}\phi'(t)\{\phi(w)-\phi(t)\}^{k-p-1}\{\phi(t)\}^p\{\gamma(t)\}^{-p}$$
$$< M_3\{\phi(w)\}^{p-k}\phi'(t)\{\phi(w)-\phi(t)\}^{k-p-1},$$

where M_3 is a constant. Hence $f_3(w,t)$ satisfies the hypotheses of Lemma 4, and so

$$\int_a^w f_3(w,t)t^{-p-1}A_{p+1}(t)\,dt \to 0 \quad \text{as} \quad w \to \infty.$$

That is, $\lim_{w\to\infty} I_{22} = 0$ in the case $r = \sigma = p$. Hence

$$\lim_{w\to\infty} I_2 = 0. \tag{3.5}$$

Finally, consider I_3, which can be written in the form

$$I_3 = \{\phi(w)\}^{-k}\int_a^w A_p(t)(w-t)^{k-p-1}\{\phi(w)-\phi(t)\}^{p+1-\rho}g(t)\,H(t)h_{p+1-\rho}(t)\,dt,$$

where

$$g(t) = \left(\frac{1}{\phi'(t)}\cdot\frac{\phi(w)-\phi(t)}{w-t}\right)^{k-p-1} \quad \text{for } a \leqq t < w, \quad g(w) = 1$$

and

$$H(t) = \prod_{\nu=1}^{p+1}\left(\frac{\{\gamma(t)\}^{\nu-1}\phi^{(\nu)}(t)}{\phi'(t)}\right)^{\beta_\nu},$$

where $\beta_1, \beta_2, \ldots, \beta_{p+1}$ are non-negative integers such that

$$1 \leqq \sum_{\nu=1}^{p+1}\beta_\nu = \rho \leqq \sum_{\nu=1}^{p+1}\nu\beta_\nu = p+1.$$

Then $H(t)$ is of bounded variation in $[a, \infty)$, because of T_A (v), and so can be expressed as the difference between two bounded monotonic non-increasing functions. Consequently, we can assume, without loss of generality, that $H(t)$ is bounded and monotonic non-increasing. Also, $\{\phi(w)-\phi(t)\}^{p+1-\rho}$, $g(t)$ and $h_{p+1-\rho}(t)$ are monotonic functions of t in the range $a \leqq t \leqq w$, the first being non-increasing since $p+1-\rho \geqq 0$ and the second non-decreasing by Lemma 5. Using the second mean-value theorem for integrals twice, we now see that

$$I_3 = \{\phi(w)\}^{-k}\{\phi(w)\}^{p+1-\rho}H(a)g(w)h_{p+1-\rho}(x)\int_{\xi_1}^{\xi_2}A_p(t)(w-t)^{k-p-1}\,dt,$$

where $w \geqq \xi_1 > \xi_2 \geqq a$, and $x = w$ or a according as $h_{p+1-\rho}(t)$ is non-decreasing or non-increasing. Hence, by Lemma 3 and (3.1),

$$I_3 = o(\{\phi(w)\}^{p+1-\rho-k}w^k h_{p+1-\rho}(x)) = o(G(w,x)), \quad \text{say}.$$

Now, by (2.2), and T_A (ii),

$$G(w,w) = O(\{\phi(w)\}^{p+1-\rho-k}\psi(w)\{\gamma(w)\}^{\rho-p-1}\{\phi'(w)\}^{k+\rho-p-1}w^k) = O(1),$$

and, by T_A (viii),

$$G(w,a) = O(\{\phi(w)\}^{p+1-\rho-k}w^k)$$
$$= O(\{\phi(w)\}^{1-\rho}) = O(1),$$

since $\rho \geqq 1$. Hence

$$\lim_{w\to\infty} I_3 = 0. \tag{3.6}$$

Because of (3.4), (3.5) and (3.6) we can deduce that (3.2) tends to a finite limit as w tends to infinity. This completes the proof of T_A.

4. Proof of T_B. Suppose that T_A(i), T_A(ii)(*a*) and T_B(vii)′ hold. It is clearly sufficient to show that T_A (viii) is a consequence.

It follows from T_B (vii)′ that, for $w \geqq a$,

$$\frac{\psi(w)\{\phi'(w)\}^{k-p}}{\{\gamma(w)\}^p} > c,$$

where c is a positive constant; and hence, by T_A (ii) (*a*),

$$\{\gamma(w)\}^p = O(\psi(w)\{\phi'(w)\}^{k-p}) = O\left(\left\{\frac{\gamma(w)}{w}\right\}^k\{\phi'(w)\}^{k-p}\right).$$

Consequently, by T_A (i),

$$w^k = O(\{\gamma(w)\phi'(w)\}^{k-p}) = O(\{w\phi'(w)\}^{k-p})$$

and so

$$w^p = O(\{\phi'(w)\}^{k-p}).$$

Hence, for $w \geqq a$, $\phi'(w) > bw^{p/(k-p)}$, where b is a positive constant, and T_A (viii) follows by integration.

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St Salvator's College
St Andrews