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ON STRONG AND ABSOLUTE SUMMABILITY

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1. Introduction. Suppose throughout that $\lambda > 0$, $\kappa > -1$, γ is real and that

$$\epsilon_n^{\gamma} = \binom{n+\gamma}{n}, \quad s_n = \sum_{r=0}^{n} a_r, \quad s_n^{\kappa} = \frac{1}{\epsilon_n^{\kappa}} \sum_{r=0}^{n} \epsilon_{n-r}^{\kappa-1} s_r \quad (n = 0, 1, \ldots).$$

The series $\sum_0^{\infty} a_n$ is said to be

(i) summable (C, κ) to s if $s_n^{\kappa} \to s$,

(ii) strongly summable $(C, \kappa+1)$ with index λ, or summable $[C, \kappa+1]_{\lambda}$, to s if

$$\frac{1}{n+1} \sum_{r=0}^{n} | s_r^{\kappa} - s |^{\lambda} = o(1),$$

(iii) absolutely summable (C, κ) with indices γ, λ, or summable $|C, \kappa, \gamma|_{\lambda}$, if

$$\sum_{n=1}^{\infty} n^{\gamma\lambda+\lambda-1} | s_n^{\kappa} - s_{n-1}^{\kappa} |^{\lambda} < \infty.$$

Definitions (ii) and (iii), for general κ, λ, γ, are due respectively to Hyslop [11] and Flett [4]. Their papers contain references to special cases considered earlier.

Let $Q = (q_{n,r})$ $(n, r = 0, 1, \ldots)$ be a (summability) matrix, and let

$$\sigma_n = Q(s_n) = \sum_{r=0}^{\infty} q_{n,r} s_r.$$

It is to be supposed that all matrices referred to in this paper are of the above type. The symbol P will be reserved for matrices $(p_{n,r})$ with $p_{n,r} \geqslant 0$ $(n, r = 0, 1, \ldots)$. The series $\sum_0^{\infty} a_n$ is said to be

(iv) summable Q to s, and we write $s_n \to s(Q)$, if σ_n is defined for all n and tends to s as $n \to \infty$.

We now generalise the above definitions of strong and absolute summability in a natural way as follows. We say that $\sum_0^{\infty} a_n$ is

(v) summable $[P, Q]_{\lambda}$ to s, and we write $s_n \to s[P, Q]_{\lambda}$, if

$$P(| \sigma_n - s |^{\lambda}) = \sum_{r=0}^{\infty} p_{n,r} | \sigma_r - s |^{\lambda}$$

is defined for each n and tends to 0 as $n \to \infty$,

(vi) summable $| Q, \gamma |_{\lambda}$ if

$$\sum_{n=1}^{\infty} n^{\gamma\lambda+\lambda-1} | \sigma_n - \sigma_{n-1} |^{\lambda} < \infty.$$

We also define " product " processes of the form QR, $[P, QR]_{\lambda}$, $| QR, \gamma |_{\lambda}$, where R is any matrix, by replacing Q in (iv), (v), (vi) by QR and taking σ_n to be $Q\{R(s_n)\}$; i.e. $\sigma_n = Q(\tau_n)$ where $\tau_n = R(s_n)$.

Denoting by C_{κ} the matrix of the transformation which changes $\{s_n\}$ into $\{s_n^{\kappa}\}$, we observe that the summability processes $[C, \kappa+1]_{\lambda}$ and $| C, \kappa, \gamma |_{\lambda}$ are respectively the same as $[C_1, C_{\kappa}]_{\lambda}$ and $| C_{\kappa}, \gamma |_{\lambda}$.

The unit matrix will be denoted by I, so that $I(s_n) = s_n$.

Let V and W be summability processes (or matrices). We shall use the notation

$$V \Rightarrow W$$

to mean that any series summable V to s is necessarily summable W to s provided that neither V nor W is an absolute summability process; otherwise we shall understand the notation to mean simply that every series summable V is also summable W. In either case we say that V is included in W. We say that V and W are equivalent and write

$$V \simeq W$$

if each is included in the other, and we write $V = W$ if V and W denote the same process (or matrix).

If $I \Rightarrow V$ and V is not an absolute summability process, then V is said to be regular.

In this paper some of the properties of the strong and absolute summability processes defined above are investigated.

2. Simple inclusions.

THEOREM 1. *If Q is any matrix and $P = (p_{n,r})$, where*

$$\sum_{r=0}^{\infty} p_{n,r} < M \quad (n = 0, 1, \ldots), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and if $\lambda > \mu > 0$, then $[P, Q]_{\lambda} \Rightarrow [P, Q]_{\mu}$.

In particular, the conclusion holds if $\lambda > \mu > 0$ and P is regular.

This generalises a result proved by Hyslop [11, Theorem 1].

Proof. By Hölder's inequality,

$$\sum_{r=0}^{\infty} p_{n,r} | w_r |^{\mu} \leqslant \left(\sum_{r=0}^{\infty} p_{n,r} | w_r |^{\lambda} \right)^{\mu/\lambda} M^{1-\mu/\lambda}$$

for any sequence $\{w_n\}$. The required inclusion follows.

To complete the proof we have only to note that (1) is a necessary condition for the regularity of P [7, Theorem 2].

Note. Here and elsewhere an inclusion involving an arbitrary matrix Q is essentially no more general than the same inclusion with I in place of Q, the former being an immediate consequence of the latter.

THEOREM 2. *If Q is any matrix and $\lambda > \mu > 0$, $\beta\lambda > \alpha\mu > 0$, then $[C_{\alpha}, Q]_{\lambda} \Rightarrow [C_{\beta}, Q]_{\mu}$.*

Proof. Let $p = \lambda/\mu$, $q = p/(p-1)$ and let $\{w_n\}$ be any sequence. Then, by Hölder's inequality (cf. Hyslop [11, Theorem 2]).

$$C_{\beta}(| w_n |^{\mu}) = \frac{1}{\epsilon_n^{\beta}} \sum_{r=0}^{n} \epsilon_r^{\beta-1} | w_{n-r} |^{\mu}$$

$$\leqslant \left\{\frac{1}{\epsilon_n^\alpha}\sum_{r=0}^{n} \epsilon_r^{\alpha-1} \mid w_{n-r}\mid^\lambda\right\}^{1/p}\left\{\frac{(\epsilon_n^\alpha)^{q/p}}{(\epsilon_n^\beta)^q}\sum_{r=0}^{n}\frac{(\epsilon_r^{\beta-1})^q}{(\epsilon_r^{\alpha-1})^{q/p}}\right\}^{1/q}$$

$$\leqslant M_1\{C_\alpha(\mid w_n\mid^\lambda)\}^{1/p}\left\{(n+1)^{\alpha q/p-\beta q}\sum_{r=0}^{n}(r+1)^{\beta q-\alpha q/p-1}\right\}^{1/q}$$

$$\leqslant M\{C_\alpha(\mid w_n\mid^\lambda)\}^{1/p}, \quad\ldots\ldots\ldots\ldots(2)$$

since $\alpha > 0$, $\beta > 0$, $\beta q - \alpha q/p = (\beta\lambda - \alpha\mu)q/\lambda > 0$. The numbers M_1 and M are independent of n and the sequence $\{w_n\}$.

The required result follows from (2).

Note. Since $C_\alpha \Rightarrow C_\beta$ $(\beta > \alpha > -1)$, it is evident that

$$[C_\alpha, Q]_\lambda \Rightarrow [C_\beta, Q]_\lambda \quad (\beta > \alpha > 0, \lambda > 0),$$

and it follows from this and a well known Tauberian theorem [7, Theorem 93] that

$$[C_\alpha, Q]_\lambda \simeq [C_1, Q]_\lambda \quad (\alpha > 1, \lambda > 0).$$

Consequently the condition $\beta\lambda > \alpha\mu > 0$ in Theorem 2 is only significant if $0 < \alpha \leqslant 1$. When $\alpha > 1$ the condition can be replaced by $\beta\lambda > \mu$.

THEOREM 3. *If P, Q are matrices and P is regular, then*

$$\text{(i) } Q \Rightarrow [P, Q]_\lambda \text{ for } \lambda > 0, \qquad \text{(ii) } [P, Q]_\lambda \Rightarrow PQ \text{ for } \lambda \geqslant 1.$$

Proof. (i) If $s_n \to s$, then, since P is regular, $P(\mid s_n - s\mid^\lambda) \to 0$, i.e. $I \Rightarrow [P, I]_\lambda$ and inclusion (i) follows.

(ii) Suppose that $s_n \to s[P, I]_\lambda$. Then, by Theorem 1, $s_n \to s[P, I]_1$ and so

$$\mid P(s_n - s)\mid \leqslant P(\mid s_n - s\mid) = o(1).$$

Since P is regular, it follows that $P(s_n) \to s$. Hence $[P, I]_\lambda \Rightarrow P$ and inclusion (ii) is an immediate consequence.

As a corollary of part (i) of Theorem 3 we have

(I). *If P, Q are regular matrices and $\lambda > 0$, then $[P, Q]_\lambda$ is regular.*

THEOREM 4. *If $\lambda \geqslant \mu > 0$, $\gamma > \delta$, then*

$$\text{(i) } \left(\sum_{n=1}^{\infty} n^{\delta\mu+\mu-1} \mid w_n\mid^\mu\right)^{1/\mu} \leqslant M\left(\sum_{n=1}^{\infty} n^{\gamma\lambda+\lambda-1}\mid w_n\mid^\lambda\right)^{1/\lambda},$$

where M is independent of the sequence $\{w_n\}$,

(ii) $\mid Q, \gamma\mid_\lambda \Rightarrow \mid Q, \delta\mid_\mu$ *for any matrix Q.*

Proof of (i). The case $\lambda = \mu$ is evident. Suppose therefore that $\lambda > \mu$. Then, by Hölder's inequality,

$$\sum_{n=1}^{\infty} n^{\delta\mu+\mu-1}\mid w_n\mid^\mu \leqslant \left(\sum_{n=1}^{\infty} n^{\gamma\lambda+\lambda-1}\mid w_n\mid^\lambda\right)^{\mu/\lambda}\left(\sum_{n=1}^{\infty} n^\alpha\right)^{1-\mu/\lambda},$$

where $\alpha(1-\mu/\lambda) = \delta\mu + \mu - 1 - (\gamma\lambda + \lambda - 1)\mu/\lambda = -\mu(\gamma - \delta) - (1 - \mu/\lambda)$, so that $\alpha < -1$. The required inequality follows.

Result (ii) is an immediate consequence of (i).

Note. The case $\lambda \geqslant \mu \geqslant 1, \gamma \geqslant 0$ of Theorem 4(i) is contained in a result proved by Flett ([4, Theorem 4]; take $\alpha = \beta$, $\tau_n^\alpha = nw_n$).

The following three results, which concern the relation of $\mid Q, \gamma\mid_\lambda$ to $\mid Q, \delta\mid_\mu$ when $\gamma = \delta$, were kindly communicated to me by Dr B. Kuttner. The first of these shows that it is not valid to replace the condition $\gamma > \delta$ by $\gamma \geqslant \delta$ in either part of Theorem 4.

A. *There are regular (and non-regular) matrices Q such that, for positive λ, μ and every γ, $\mid Q, \gamma\mid_\lambda$ is not included in $\mid Q, \gamma\mid_\mu$ unless $\lambda = \mu$.*

B. *There are regular (and non-regular) matrices Q such that, for every γ, $\mid Q, \gamma\mid_\lambda \Rightarrow \mid Q, \gamma\mid_\mu$ whenever $\lambda > \mu > 0$.*

C. *If $\lambda > \mu > \nu > 0$ and Q is any matrix, then every series summable $\mid Q, \gamma\mid_\lambda$ and $\mid Q, \gamma\mid_\nu$ is also summable $\mid Q, \gamma\mid_\mu$.*

Proofs. A. Suppose that $Q = (q_{n,r})$ is a matrix having the property that given any sequence $\{\sigma_n\}$ there is a sequence $\{s_n\}$ (not necessarily unique) satisfying the equations

$$\sigma_n = Q(s_n) = \sum_{r=0}^{\infty} q_{n,r}s_r \quad (n = 0, 1, \ldots).$$

In particular, Q could be any matrix with $q_{n,r} = 0$ for $r > n$, $q_{n,n} \neq 0$ $(n = 0, 1, \ldots)$.

Let $\alpha > 0$; and let $x_1 = x_2 = 0$,

$$x_n = n^{-1}(\log n)^{-1/\lambda}(\log\log n)^{-1/\lambda-\alpha} \quad \text{for } n \geqslant 3,$$

$$y_n = \begin{cases} m^{-1/\lambda-\alpha}2^{-m(1-1/\lambda)} & \text{for } n = 2^m \quad (m = 0, 1, \ldots),\\ 0 & \text{otherwise.}\end{cases}$$

Then $\sum_{n=1}^{\infty}(x_n)^\mu n^{\mu-1}$ is convergent if and only if $\mu \geqslant \lambda$ and $\sum_{n=1}^{\infty}(y_n)^\mu n^{\mu-1}$ is convergent if and only if $\mu \leqslant \lambda$. Hence $\sum_{n=1}^{\infty}(x_n + y_n)^\mu n^{\mu-1}$ is convergent if and only if $\mu = \lambda$.

Now let $\{\sigma_n\}$, $\{s_n\}$ be sequences such that

$$n^\gamma(\sigma_n - \sigma_{n-1}) = x_n + y_n \quad (n \geqslant 1)$$

and $Q(s_n) = \sigma_n$. The series of which $\{s_n\}$ is the sequence of partial sums is then summable $\mid Q, \gamma\mid_\lambda$ but not $\mid Q, \gamma\mid_\mu$ for any $\mu \neq \lambda$. Result A follows.

B. Given an arbitrary matrix $Q = (q_{n,r})$, form the matrix $Q^* = (q^*_{n,r})$ by repeating certain rows in Q as follows: let

$$q^*_{0,r} = q_{0,r}, \quad q^*_{n,r} = q_{m,r} \quad \text{for } 2^{m-1} \leqslant n < 2^m \quad (m = 1, 2, \ldots).$$

Note that Q^* is regular if and only if Q is regular.

Let $s_n = \sum_{r=0}^{n} a_r$, $\sigma_n^* = Q^*(s_n)$ and let

$$\delta_m = \sigma^*_{2^m} - \sigma^*_{2^m-1} \quad (m = 0, 1, \ldots).$$

Then $\sigma_n^* - \sigma_{n-1}^* = 0$ when $n \neq 2^m$ and so summability $\mid Q^*, \gamma\mid_\lambda$ of $\sum_0^\infty a_n$ is equivalent to the convergence of

$$\sum_{m=0}^{\infty} 2^{m(\gamma\lambda+\lambda-1)}\mid \delta_m\mid^\lambda.$$

Consequently, if $\sum_0^\infty a_n$ is summable $|Q^*, \gamma|_\lambda$, then

$$\delta_m = o(2^{-m(\gamma+1-1/\lambda)})$$

and so
$$2^{m(\gamma\mu+\mu-1)} |\delta_m|^\mu = o(2^{-m(1-\mu/\lambda)}),$$

from which it follows that the series is summable $|Q^*, \gamma|_\mu$ provided $\lambda > \mu > 0$. i.e. $|Q^*, \gamma|_\lambda \Rightarrow |Q^*, \gamma|_\mu$ for $\lambda > \mu > 0$.

C. If $\lambda > \mu > \nu > 0$ and $\{w_n\}$ is any sequence, then, by Hölder's inequality,

$$\left(\sum_{n=1}^\infty n^{\gamma\mu+\mu-1} |w_n|^\mu\right)^{\lambda-\nu} \leqslant \left(\sum_{n=1}^\infty n^{\gamma\lambda+\lambda-1} |w_n|^\lambda\right)^{\mu-\nu} \left(\sum_{n=1}^\infty n^{\gamma\nu+\nu-1} |w_n|^\nu\right)^{\lambda-\mu};$$

and the required " convexity " result is a direct consequence.

3. Hausdorff matrices. Given a real sequence $\{\xi_n\}$, let

$$x_{n,r} = \begin{cases} \binom{n}{r} \sum_{\nu=0}^{n-r} (-1)^\nu \binom{n-r}{\nu} \xi_{r+\nu} & \text{for } 0 \leqslant r \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

and denote the matrix $(x_{n,r})$ by (h, ξ_n). Matrices of this type are said to be real Hausdorff matrices. We shall assume hereafter that all Hausdorff matrices considered are real.

Let $X = (h, \xi_n)$, $Y = (h, \eta_n)$. Then it is known that $XY = YX = (h, \xi_n\eta_n)$. Consequently $X^{-1} = (h, 1/\xi_n)$ provided $\xi_n \neq 0$, and it is familiar and easily verified that in this case $X \Rightarrow Y$ if and only if YX^{-1} is regular.

Further, it is known that X is regular if and only if

$$\xi_n = \int_0^1 t^n \, d\chi(t),$$

where χ is a real function of bounded variation in [0, 1] such that

$$\chi(0+) = \chi(0) = \chi(1) - 1, \quad \text{......(3)}$$

it being assumed in the case of ξ_0 that $0^0 = 1$.

The above results are proved in [7, Ch. XI].

Suppose as before that $s_n = \sum_{r=0}^n a_r$ and let $\sigma_n = X(s_n)$, $\sigma_{-1} = 0$. Since both X and C_1^{-1} are Hausdorff matrices [7, § 11.2],

$$XC_1^{-1}(s_n) = C_1^{-1}X(s_n). \quad \text{......(4)}$$

Also, it is easily verified that

$$C_1^{-1}(s_n) = s_n + na_n.$$

Consequently

$$\sigma_n + X(na_n) = X(s_n + na_n) = XC_1^{-1}(s_n) = C_1^{-1}X(s_n) = C_1^{-1}(\sigma_n) = \sigma_n + n(\sigma_n - \sigma_{n-1}),$$

and so

$$X(na_n) = n(\sigma_n - \sigma_{n-1}) \quad (n = 1, 2, \ldots). \quad \text{......(5)}$$

Conversely, reversing the above argument, we see that (4) holds for any matrix X satisfying (5), and it is known [7, Theorem 198] that (4) implies that X must be a Hausdorff matrix.

It follows from (5) that, for a Hausdorff matrix X, $\sum_0^\infty a_n$ is summable $|X, \gamma|_\lambda$ if and only if

$$\sum_{n=1}^\infty n^{\gamma\lambda-1} |X(na_n)|^\lambda < \infty.$$

We proceed to prove two general theorems about strong and absolute summability processes associated with Hausdorff matrices. We shall use

Lemma 1. *If $X = (h, \xi_n)$, $\bar{X} = (h, \bar{\xi}_n)$, where*

$$\xi_n = \int_0^1 t^n \, d\chi(t), \quad \bar{\xi}_n = \int_0^1 t^n \, |d\chi(t)| < \infty \quad (n = 0, 1, \ldots),$$

and if $\lambda \geqslant 1$, then, for any sequence $\{w_n\}$,

$$|X(w_n)|^\lambda \leqslant (\bar{\xi}_0)^{\lambda-1}\bar{X}(|w_n|^\lambda).$$

Proof. Let $X = (x_{n,r})$, $\bar{X} = (\bar{x}_{n,r})$. Then it is known and easily verified that, for $0 \leqslant r \leqslant n$,

$$x_{n,r} = \binom{n}{r} \int_0^1 t^r(1-t)^{n-r} \, d\chi(t), \quad \bar{x}_{n,r} = \binom{n}{r} \int_0^1 t^r(1-t)^{n-r} \, |d\chi(t)|.$$

Hence, by Hölder's inequality,

$$|X(w_n)|^\lambda = \left|\sum_{r=0}^n x_{n,r}w_r\right|^\lambda \leqslant \left(\sum_{r=0}^n \bar{x}_{n,r}\right)^{\lambda-1} \sum_{r=0}^n \bar{x}_{n,r} |w_r|^\lambda = (\bar{\xi}_0)^{\lambda-1}\bar{X}(|w_n|^\lambda).$$

Theorem 5. *If P, X are regular Hausdorff matrices, Q is any matrix and $\lambda \geqslant 1$, then $[P, Q]_\lambda \Rightarrow [P, XQ]_\lambda$.*

Proof. Let $X = (h, \xi_n)$ and let $\sigma_n = X(s_n)$. Since X is regular,

$$\sigma_n - s = X(s_n - s),$$

and

$$\xi_n = \int_0^1 t^n \, d\chi(t)$$

where χ is a real function of bounded variation in [0, 1] satisfying (3). Hence, using Lemma 1 and its notation, we get

$$|\sigma_n - s|^\lambda \leqslant (\bar{\xi}_0)^{\lambda-1} \bar{X}(|s_n - s|^\lambda).$$

Since P is a Hausdorff matrix with non-negative elements and $\bar{X}$ is a Hausdorff matrix, it follows that

$$P(|\sigma_n - s|^\lambda) \leqslant (\bar{\xi}_0)^{\lambda-1}P\bar{X}(|s_n - s|^\lambda) = (\bar{\xi}_0)^{\lambda-1}\bar{X}P(|s_n - s|^\lambda). \quad \text{......(6)}$$

Now it is easily verified by means of a variant of Toeplitz's theorem [7, Theorem 4] that $\bar{X}$, though not necessarily regular, is such that $\bar{X}(u_n) \to 0$ whenever $u_n \to 0$. Hence if $P(|s_n - s|^\lambda) \to 0$ then, by (6), $P(|\sigma_n - s|^\lambda) \to 0$, i.e. $[P, I]_\lambda \Rightarrow [P, X]_\lambda$. The required inclusion follows.

As an immediate consequence of the above theorem we have

(II). *If $\lambda \geqslant 1$ and P, Y, Z are Hausdorff matrices such that P is regular, $Y = (h, \eta_n)$ with $\eta_n \neq 0$, and $Y \Rightarrow Z$, then $[P, Y]_\lambda \Rightarrow [P, Z]_\lambda$.*

THEOREM 6. *If* $X = (h, \xi_n)$, *where*

$$\xi_n = \int_0^1 t^n \, d\chi(t) \quad (n = 0, 1, \ldots),$$

χ *being a real function of bounded variation in* $[0, 1]$, *and if*

$$\int_0^1 t^{-\gamma} \, | \, d\chi(t) \, | < \infty \qquad \text{(7)}$$

and $\lambda \geqslant 1$, *then*

$$\text{(i)} \sum_{n=1}^{\infty} n^{\gamma\lambda - 1} \, | \, X(na_n) \, |^{\lambda} \leqslant M \sum_{n=1}^{\infty} n^{\gamma\lambda-1} \, | \, na_n \, |^{\lambda},$$

where M *is independent of the sequence* $\{a_n\}$,

$$\text{(ii)} \ | \, Q, \gamma \, |_{\lambda} \Rightarrow | \, XQ, \gamma \, |_{\lambda} \quad \textit{for any matrix } Q.$$

When $\gamma > 0$ the integral in condition (7) should be interpreted in the Lebesgue-Stieltjes sense; when $\gamma \leqslant 0$ the condition is redundant.

Proof of (i). Suppose first that $\gamma \leqslant 0$. Then, by Lemma 1, since $n^{\gamma\lambda} \leqslant r^{\gamma\lambda}$ for $n \geqslant r$,

$$\sum_{n=1}^{\infty} n^{\gamma\lambda-1} \, | \, X(na_n) \, |^{\lambda} \leqslant (\xi_0)^{\lambda-1} \sum_{n=1}^{\infty} n^{\gamma\lambda-1} \sum_{r=1}^{n} \, | \, ra_r \, |^{\lambda} \binom{n}{r} \int_0^1 t^r (1-t)^{n-r} \, | \, d\chi(t) \, |$$

$$= (\xi_0)^{\lambda-1} \int_0^1 | \, d\chi(t) \, | \sum_{r=1}^{\infty} r^{-1} \, | \, ra_r \, |^{\lambda} \, t^r \sum_{n=r}^{\infty} n^{\gamma\lambda} \binom{n-1}{r-1} (1-t)^{n-r}$$

$$\leqslant (\xi_0)^{\lambda} \sum_{r=1}^{\infty} r^{\gamma\lambda-1} \, | \, ra_r \, |^{\lambda},$$

as required.

Suppose now that $\gamma > 0$, and let

$$f_n(t) = \sum_{r=0}^{n} \binom{n}{r} t^r (1-t)^{n-r} ra_r,$$

where $0 \leqslant t \leqslant 1$. Then (cf. Hardy [7, § 11.17]), by Hölder's inequality,

$$| \, f_n(t) \, |^{\lambda} \leqslant \sum_{r=0}^{n} \binom{n}{r} t^r (1-t)^{n-r} \, | \, ra_r \, |^{\lambda} \left\{ \sum_{r=0}^{n} \binom{n}{r} t^r (1-t)^{n-r} \right\}^{\lambda-1}$$

$$= \sum_{r=1}^{n} \binom{n}{r} t^r (1-t)^{n-r} \, | \, ra_r \, |^{\lambda},$$

and so

$$\sum_{n=1}^{\infty} n^{\gamma\lambda-1} \, | \, f_n(t) \, |^{\lambda} \leqslant M_1 \sum_{n=1}^{\infty} \epsilon_n^{\gamma\lambda-1} \sum_{r=1}^{n} \binom{n}{r} t^r (1-t)^{n-r} \, | \, ra_r \, |^{\lambda}$$

$$= M_1 \sum_{r=1}^{\infty} \epsilon_r^{\gamma\lambda-1} \, | \, ra_r \, |^{\lambda} \, t^r \sum_{n=r}^{\infty} \epsilon_{n-r}^{\gamma\lambda+r-1} (1-t)^{n-r}$$

$$\leqslant M_2 t^{-\gamma\lambda} \sum_{r=1}^{\infty} r^{\gamma\lambda-1} \, | \, ra_r \, |^{\lambda},$$

where M_1 and M_2 are independent of $\{a_n\}$.

Now
$$X(na_n) = \int_0^1 f_n(t) \, d\chi(t)$$

and so, by a form of Minkowski's inequality,

$$\left(\sum_{n=1}^{\infty} n^{\gamma\lambda-1} \, | \, X(na_n) \, |^{\lambda} \right)^{1/\lambda} \leqslant \int_0^1 | \, d\chi(t) \, | \left(\sum_{n=1}^{\infty} n^{\gamma\lambda-1} \, | \, f_n(t) \, |^{\lambda} \right)^{1/\lambda}$$

$$\leqslant M_2^{1/\lambda} \int_0^1 t^{-\lambda} \, | \, d\chi(t) \, | \left(\sum_{r=1}^{\infty} r^{\gamma\lambda-1} \, | \, ra_r \, |^{\lambda} \right)^{1/\lambda}.$$

The proof of part (i) is thus complete.

It follows from (i) that $| \, I, \gamma \, |_{\lambda} \Rightarrow | \, X, \gamma \, |_{\lambda}$, and inclusion (ii) is an immediate consequence.

The next theorem generalises a result given by Hyslop [**11**, Theorem 3].

THEOREM 7. *If* P *is a regular matrix,* Q *is a matrix and* $\lambda \geqslant 1$, *then necessary and sufficient conditions for a series to be summable* $[P, Q]_{\lambda}$ *to* s *are that it be summable* PQ *to* s *and summable* $[P, (I-P)Q]_{\lambda}$ *to* 0.

Proof. Let $\sigma_n = Q(s_n)$, $\tau_n = P(\sigma_n)$. We have to prove that

$$P(\, | \, \sigma_n - s \, |^{\lambda}) = o(1) \qquad \text{(8)}$$

if and only if

$$\tau_n \to s \qquad \text{(9)}$$

and

$$P(\, | \, \sigma_n - \tau_n \, |^{\lambda}) = o(1). \qquad \text{(10)}$$

(i) Suppose that (8) holds. Then, by Theorem 3(ii), (9) holds, and so $P(\, | \, \tau_n - s \, |^{\lambda}) = o(1)$ since P is regular. Hence, by Minkowski's inequality and (8),

$$\{P(\, | \, \sigma_n - \tau_n \, |^{\lambda})\}^{1/\lambda} \leqslant \{P(\, | \, \sigma_n - s \, |^{\lambda})\}^{1/\lambda} + \{P(\, | \, \tau_n - s \, |^{\lambda})\}^{1/\lambda} = o(1)$$

and (10) follows.

(ii) Suppose that (9) and (10) hold. Since P is regular, it follows from (9) that

$$P(\, | \, \tau_n - s \, |^{\lambda}) = o(1).$$

Hence, by Minkowski's inequality and (10),

$$\{P(\, | \, \sigma_n - s \, |^{\lambda})\}^{1/\lambda} \leqslant \{P(\, | \, \sigma_n - \tau_n \, |^{\lambda})\}^{1/\lambda} + \{P(\, | \, \tau_n - s \, |^{\lambda})\}^{1/\lambda} = o(1)$$

so that (8) holds.

The proof is thus complete.

Now it is known [7, Ch. XI] that $C_{\kappa} = (h, 1/\epsilon_n^{\kappa})$ $(\kappa > -1)$ and that

$$C_{\alpha} C_{\beta} \simeq C_{\alpha+\beta} \quad (\alpha > -1, \beta > -1, \alpha+\beta > -1). \qquad \text{(11)}$$

Further, if $s_n = \sum_{r=0}^{n} a_r$, then for any Hausdorff matrix X,

$$(I - C_1) X(s_n) = X(I - C_1)(s_n) = X\{s_n - C_1(s_n)\} = XC_1(na_n). \qquad \text{(12)}$$

In virtue of (12) we have the following corollary of Theorem 7.

(III). *If* X *is a Hausdorff matrix and* $\lambda \geqslant 1$, *then necessary and sufficient conditions for a series* $\sum_0^{\infty} a_n$ *to be summable* $[C_1, X]_{\lambda}$ *to* s *are that it be summable* C_1X *to* s *and that*

$$na_n \to 0 \, [C_1, C_1X]_{\lambda}.$$

Now by (11), $C_1C_{\alpha-1} \simeq C_\alpha$ $(\alpha > 0)$, and so, by result (II), $[C_1, C_1C_{\alpha-1}]_\lambda \simeq [C_1, C_\alpha]_\lambda$ $(\alpha > 0, \lambda \geqslant 1)$. Consequently, by (III), we have

(IV). *If* $\lambda \geqslant 1$, $\alpha > 0$, *then necessary and sufficient conditions for a series* $\sum_0^\infty a_n$ *to be summable* $[C, \alpha]_\lambda$ *to s are that it be summable* (C, α) *to s and that* $\sum_{n=0}^m | C_\alpha(na_n) |^\lambda = o(m)$.

This result has been proved directly by Hyslop [**11**] and it suggested the following definition of summability $[C, 0]_\lambda$ to him : $\sum_0^\infty a_n$ is summable $[C, 0]_\lambda$ to s if it is convergent with sum s and

$$\sum_{n=0}^m | na_n |^\lambda = o(m).$$

4. Equivalence of Cesàro and Hölder summability processes. For any real α let H_α be the Hausdorff matrix $(h, (n+1)^{-\alpha})$. Then $C_1 = H_1$, $H_\alpha H_\beta = H_{\alpha+\beta}$, and it is known [**7**, Theorem 211] that

$$C_\kappa \simeq H_\kappa \quad (\kappa > -1). \tag{13}$$

In conformity with the notation described in § 1, we denote the Hölder type summability processes H_α, $[H_1, H_{\alpha-1}]_\lambda$ and $| H_\alpha, \gamma |_\lambda$ by (H, α), $[H, \alpha]_\lambda$ and $| H, \alpha, \gamma |_\lambda$ respectively.

We now prove two theorems.

THEOREM 8. *If* $\alpha \geqslant 0$, $\lambda \geqslant 1$, *then* $[C, \alpha]_\lambda \simeq [H, \alpha]_\lambda$.

For $\alpha > 0$ this follows from (13) by result (II), and for $\alpha = 0$ it is a consequence of (III) with $X = H_{-1} = C_1^{-1}$.

The next theorem is a generalisation of the known result (see Knopp and Lorentz [**12**] and Morley [**14**]) that

$$| C, \alpha, 0 |_1 \simeq | H, \alpha, 0 |_1 \qquad (\alpha > -1).$$

THEOREM 9. (i) *If* $\alpha > -1$, $\lambda \geqslant 1$, $\gamma < \min(1, 1+\alpha)$, *then*

$$| C, \alpha, \gamma |_\lambda \Rightarrow | H, \alpha, \gamma |_\lambda.$$

(ii) *If either* $\alpha > -1$, $\lambda \geqslant 1$, $\gamma < 1$ *or* $\alpha = 2, 3, \ldots,$ $\lambda \geqslant 1$, $\gamma < 2$, *then*

$$| H, \alpha, \gamma |_\lambda \Rightarrow | C, \alpha, \gamma |_\lambda.$$

In connection with the second part of (ii) it should be noted that

$$| H, 0, \gamma |_\lambda = | C, 0, \gamma |_\lambda \quad \text{and} \quad | H, 1, \gamma |_\lambda = | C, 1, \gamma |_\lambda.$$

The cases $\gamma \leqslant 0$ of the propositions contained in Theorem 9 follow directly from (13) by Theorem 6(ii). To deal with the remaining cases we shall use

LEMMA 2. *If* $\sigma_0 < 0$ *and* $g(s)$ *is an analytic function of* $s = \sigma + i\tau$ *in the region* $\sigma > \sigma_0$, *and if, for* $\sigma > \sigma_0$ *and large* $| s |$,

$$g(s) = K + O(| s |^{-\delta}),$$

where K, δ *are constants and* $\delta > \frac{1}{2}$, *then*

$$g(n) = \int_0^1 t^n \, d\chi(t) \quad (n \geqslant 0),$$

where χ *is a function of bounded variation in* [0, 1] *such that*

$$\int_0^1 t^c \, | d\chi(t) | < \infty$$

for every $c > \sigma_0$.

Proof. Let $f(s) = g(s) - K$. Then, for $c > \sigma_0 + \epsilon > \sigma_0$,

$$\int_{-\infty}^{\infty} | f(c+it) |^2 \, dt < M_\epsilon,$$

where M_ϵ is a finite number independent of c. Hence, by a result due to Rogosinski [**15**, 185–6],

$$f(n) = \int_0^1 t^n \phi(t) \, dt \quad (n \geqslant 0),$$

where $t^c \phi(t) \in L(0, 1)$ for every $c > \sigma_0 + \epsilon$ and so for every $c > \sigma_0$.

Consequently

$$g(n) = \int_0^1 t^n \, d\chi(t) \quad (n \geqslant 0),$$

where
$$\chi(t) = \int_0^t \phi(u) \, du \text{ for } 0 \leqslant t < 1 \text{ and } \chi(1) = K + \int_0^1 \phi(u) \, du.$$

It is evident that $\int_0^1 t^c \, | d\chi(t) | < \infty$ for every $c > \sigma_0$.

The lemma is thus proved.

Completion of the proof of Theorem 9. Let

$$w(s) = (s+1)^{-\alpha} \frac{\Gamma(s+\alpha+1)}{\Gamma(\alpha+1)\Gamma(s+1)}$$

and let W be the Hausdorff matrix (h, w_n), where $w_n = w(n)$.

(i) By Stirling's theorem, $w(s)$ satisfies the hypotheses of $g(s)$ in Lemma 2 with $\delta = 1$, $\sigma_0 = \max(-1, -1-\alpha)$. Hence by Theorem 6 (ii), with $X = W$,

$$| C_\alpha, \gamma |_\lambda \Rightarrow | WC_\alpha, \gamma |_\lambda$$

for $-\gamma > \sigma_0$, i.e. for $\gamma < \min(1, 1+\alpha)$. Since $WC_\alpha = H_\alpha$, the proof of part (i) is complete.

(ii) The function $1/w(s)$ satisfies the hypotheses of $g(s)$ in Lemma 2 with $\delta = 1$, $\sigma_0 = -1$ when $\alpha > -1$ and with $\delta = 1$, $\sigma_0 = -2$ when $\alpha = 2, 3, \ldots$. Hence by Theorem 6(ii), with $X = W^{-1}$,

$$| H_\alpha, \gamma |_\lambda \Rightarrow | W^{-1}H_\alpha, \gamma |_\lambda$$

for $-\gamma > -1$ when $\alpha > -1$, and for $-\gamma > -2$ when $\alpha = 2, 3, \ldots$. Since $W^{-1}H_\alpha = C_\alpha$, this completes the proof of part (ii).

5. Hausdorff matrices associated with functions of class L^p. In this section we deal with Hausdorff matrices (h, ξ_n) such that $\xi_n = \int_0^1 t^n \phi(t) \, dt$, where $\phi(t) \in L(0, 1)$ and $t^c \phi(t) \in L^p(0, 1)$ for some real c and some $p > 1$. It is known [**7**, Theorem 215] that a Hausdorff matrix $(x_{n,r})$ satisfies these conditions with $c = 0$ if and only if

$$\sum_{r=0}^{n} |x_{n,r}|^p < M(n+1)^{1-p} \quad (n = 0, 1, \ldots),$$

where M is independent of n. Note that if $\phi(t)$ is in $L^p(0, 1)$ then it is necessarily in $L(0, 1)$.

We establish two theorems which augment Theorems 5 and 6. In the proof of the first of these we use

LEMMA 3. *Let $\phi(t)$ be a real function in the class $L^p(0, 1)$, where $p > 1$, and let*

$$\xi_n = \int_0^1 t^n \phi(t)\, dt, \quad \xi_n^{(p)} = \int_0^1 t^n |\phi(t)|^p\, dt \quad (n = 0, 1, \ldots), \quad X = (h, \xi_n), \quad X^{(p)} = (h, \xi_n^{(p)}).$$

If $\mu > \lambda \geqslant 1$ and $1 + 1/\mu - 1/\lambda = 1/p$, then, for any sequence $\{w_n\}$,

$$|X(w_n)|^\mu \leqslant (\xi_0^{(p)})^{\mu(1-1/\lambda)} \{C_1(|w_n|^\lambda)\}^{\mu/\lambda - 1} X^{(p)}(|w_n|^\lambda).$$

Proof. Let

$$f_n(t) = \sum_{r=0}^{n} \binom{n}{r} t^r (1-t)^{n-r} w_r,$$

where $0 \leqslant t \leqslant 1$. Then, as in the proof of Theorem 6,

$$|f_n(t)|^\lambda \leqslant \sum_{r=0}^{n} \binom{n}{r} t^r (1-t)^{n-r} |w_r|^\lambda,$$

so that

$$\int_0^1 |f_n(t)|^\lambda\, dt \leqslant \frac{1}{n+1} \sum_{r=0}^{n} |w_r|^\lambda = C_1(|w_n|^\lambda) \quad \ldots\ldots(14)$$

and

$$\int_0^1 |\phi(t)|^p |f_n(t)|^\lambda\, dt \leqslant X^{(p)}(|w_n|^\lambda). \quad \ldots\ldots(15)$$

Further, using Hölder's inequality twice, we have

$$\begin{aligned} |X(w_n)|^\lambda &= \left| \int_0^1 \phi(t) f_n(t)\, dt \right|^\lambda \\ &\leqslant \left(\int_0^1 |\phi(t)|^{p(1-1/\lambda)} |\phi(t)|^{p/\mu} |f_n(t)|\, dt \right)^\lambda \\ &\leqslant \left(\int_0^1 |\phi(t)|^p\, dt \right)^{\lambda-1} \int_0^1 |\phi(t)|^{p\lambda/\mu} |f_n(t)|^\lambda\, dt \\ &\leqslant (\xi_0^{(p)})^{\lambda-1} \left(\int_0^1 |f_n(t)|^\lambda\, dt \right)^{1-\lambda/\mu} \left(\int_0^1 |\phi(t)|^p |f_n(t)|^\lambda\, dt \right)^{\lambda/\mu}. \end{aligned} \quad \ldots\ldots(16)$$

The required result follows from (14), (15) and (16).

THEOREM 10. *Let $\mu > \lambda \geqslant 1$, $1/p = 1 + 1/\mu - 1/\lambda$, and let $X = (h, \xi_n)$, where*

$$\xi_n = \int_0^1 t^n \phi(t)\, dt \quad \textit{with } \phi(t) \in L^p(0, 1) \textit{ and } \xi_0 = 1.$$

Then $[C_1, Q]_\lambda \Rightarrow [C_1, XQ]_\mu$ for any matrix Q.

Proof. Observe that X is a regular Hausdorff matrix and (in the notation of Lemma 3) that $X^{(p)}$ is a Hausdorff matrix such that $X^{(p)}(v_n) \to 0$ whenever $v_n \to 0$.

Suppose that $s_n \to s[C_1, Q]_\lambda$, and let

$$w_n = Q(s_n) - s = \sigma_n - s, \quad v_n = C_1(|w_n|^\lambda), \quad k = (\xi_0^{(p)})^{\mu(1-1/\lambda)} \sup_{n \geq 0} (v_n)^{\mu/\lambda - 1}.$$

Then $v_n \to 0$ so that k is finite and, by Lemma 3,

$$C_1(|X(\sigma_n) - s|^\mu) = C_1(|X(w_n)|^\mu) \leqslant k C_1 X^{(p)}(|w_n|^\lambda) = k X^{(p)}(v_n) = o(1).$$

Hence $s_n \to s[C_1, XQ]_\mu$, and the theorem is established.

Remark. I am indebted to Dr B. Kuttner for pointing out that Theorem 10 continues to hold when $\mu = \infty$ (with $1/p = 1 - 1/\lambda$ if $\lambda > 1$ and $p = \infty$ if $\lambda = 1$) provided the following natural conventions are taken to apply : (i) $[C_1, XQ]_\infty$ denotes the same summability process as XQ (cf. Glatfeld [**6**, Theorem 4]), (ii) $\phi(t) \in L^\infty(0, 1)$ means that $\phi(t)$ is measurable and essentially bounded in $(0, 1)$. To justify this assertion suppose that the hypotheses of Theorem 10 hold with $\mu = \infty$. Then (16) can be replaced by the simpler inequality

$$|X(w_n)|^\lambda \leqslant m \int_0^1 |f_n(t)|^\lambda\, dt,$$

where $m = \left(\int_0^1 |\phi(t)|^p\, dt \right)^{\lambda-1}$ if $\lambda > 1$ and $m = \operatorname*{ess\text{-}sup}_{0<t<1} |\phi(t)|$ if $\lambda = 1$. Since (14) applies unchanged, it follows that

$$|X(w_n)|^\lambda \leqslant m C_1(|w_n|^\lambda);$$

and this yields the required inclusion, namely $|C_1, Q|_\lambda \Rightarrow XQ$.

THEOREM 11. *Let $\mu > \lambda \geqslant 1$, $1/p = 1 + 1/\mu - 1/\lambda$, $\gamma \geqslant 0$, and let $X = (h, \xi_n)$, where $\xi_n = \int_0^1 t^n \phi(t)\, dt$ with $\phi(t) \in L(0, 1)$ and $t^{1-\gamma-1/p}\phi(t) \in L^p(0, 1)$.*

Then

$$\text{(i)} \quad \left(\sum_{n=1}^{\infty} n^{\gamma\mu - 1} |X(na_n)|^\mu \right)^{1/\mu} \leqslant M \left(\sum_{n=1}^{\infty} n^{\gamma\lambda - 1} |na_n|^\lambda \right)^{1/\lambda},$$

where M is independent of the sequence $\{a_n\}$,

$$\text{(ii)} \quad |Q, \gamma|_\lambda \Rightarrow |XQ, \gamma|_\mu \textit{ for any matrix } Q.$$

Proof of (i). We shall use the symbols M_1, M_2, M_3, M_4 to denote positive numbers independent of n, t and the sequence $\{a_n\}$.

Let

$$S = \sum_{n=1}^{\infty} n^{\gamma\lambda - 1} |na_n|^\lambda < \infty,$$

and let

$$f_n(t) = \sum_{r=0}^{n} \binom{n}{r} t^r (1-t)^{n-r} r a_r,$$

where $0 \leqslant t \leqslant 1$. Then, as before,

$$|f_n(t)|^\lambda \leqslant \sum_{r=0}^{n} \binom{n}{r} t^r (1-t)^{n-r} |r a_r|^\lambda,$$

and so

$$n^{\gamma\lambda}\int_0^1 t^{\gamma\lambda-1}\,|f_n(t)|^\lambda\,dt \leqslant n^{\gamma\lambda}\sum_{r=1}^{n}|ra_r|^\lambda\binom{n}{r}\int_0^1 t^{\gamma\lambda+r-1}(1-t)^{n-r}\,dt$$

$$= \frac{n^{\gamma\lambda}}{\epsilon_n^{\gamma\lambda}}\sum_{r=1}^{n} r^{-1}\epsilon_{r-1}^{\gamma\lambda}\,|ra_r|^\lambda$$

$$\leqslant M_1\sum_{r=1}^{n} r^{\gamma\lambda-1}\,|ra_r|^\lambda = M_1 S. \quad \text{..........................(17)}$$

Also

$$\sum_{n=1}^{\infty} n^{\gamma\lambda-1}\,|f_n(t)|^\lambda \leqslant M_2 t^{-\gamma\lambda}\sum_{r=1}^{\infty} r^{\gamma\lambda-1}\,|ra_r|^\lambda = M_2 S t^{-\gamma\lambda}; \quad \text{.................(18)}$$

for $\gamma > 0$ this has been established in the proof of Theorem 6 (i), and an argument similar to that used in the proof of the case $\gamma = 0$ of Theorem 6 (i), involving the identity

$$\frac{1}{n}\binom{n}{r} = \frac{1}{r}\binom{n-1}{r-1},$$

shows that the inequality is valid when $\gamma = 0$.

Now let $c = 1-\gamma-1/p$, $\psi(t) = t^c\phi(t)$, and let

$$k = \int_0^1 |\psi(t)|^p\,dt.$$

Then k is finite, and, as in the proof of Lemma 3,

$$|X(na_n)|^\lambda = \left|\int_0^1 \psi(t)\,t^{-c}f_n(t)\,dt\right|^\lambda$$

$$\leqslant k^{\lambda-1}\int_0^1 |\psi(t)|^{p\lambda/\mu}t^{1-\lambda c-\lambda\gamma}t^{\gamma\lambda-1}\,|f_n(t)|^\lambda\,dt$$

$$\leqslant k^{\lambda-1}\left(\int_0^1 t^{\gamma\lambda-1}\,|f_n(t)|^\lambda\,dt\right)^{1-\lambda/\mu}\left(\int_0^1 |\psi(t)|^p t^{\mu/\lambda-\mu c-\mu\gamma}t^{\gamma\lambda-1}\,|f_n(t)|^\lambda\,dt\right)^{\lambda/\mu}$$

$$= k^{\lambda-1}\left(\int_0^1 t^{\gamma\lambda-1}\,|f_n(t)|^\lambda\,dt\right)^{1-\lambda/\mu}\left(\int_0^1 |\psi(t)|^p t^{\gamma\lambda}\,|f_n(t)|^\lambda\,dt\right)^{\lambda/\mu},$$

since $\mu/\lambda-\mu c-\mu\gamma = \mu/\lambda-\mu(1-1/p) = 1$. Hence

$$n^{\gamma\mu-1}\,|X(na_n)|^\mu \leqslant k^{(\lambda-1)\mu/\lambda}\left(n^{\gamma\lambda}\int_0^1 t^{\gamma\lambda-1}\,|f_n(t)|^\lambda\,dt\right)^{\mu/\lambda-1}\int_0^1 |\psi(t)|^p t^{\gamma\lambda}n^{\gamma\lambda-1}\,|f_n(t)|^\lambda\,dt$$

and so, by (17) and (18),

$$\sum_{n=1}^{\infty} n^{\gamma\mu-1}\,|X(na_n)|^\mu \leqslant M_3 S^{\mu/\lambda-1}\int_0^1 |\psi(t)|^p t^{\gamma\lambda}\,dt\sum_{n=1}^{\infty} n^{\gamma\lambda-1}\,|f_n(t)|^\lambda$$

$$\leqslant M_3 S^{\mu/\lambda-1}kM_2 S = M_4 S^{\mu/\lambda}.$$

Result (i) follows. Hence $|I,\gamma|_\lambda \Rightarrow |X,\gamma|_\mu$, and result (ii) is an immediate consequence.

We state next two propositions.

(V). *If Q is any matrix and either* (i) $\mu \geqslant \lambda \geqslant 1$, $\rho > 1/\lambda-1/\mu$ *or* (ii) $\mu > \lambda > 1$, $\rho = 1/\lambda-1/\mu$, *then*

$$[C_1, Q]_\lambda \Rightarrow [C_1, C_\rho Q]_\mu.$$

(VI). *If Q is any matrix and either* (i) $\mu \geqslant \lambda \geqslant 1$, $\rho > 1/\lambda-1/\mu$, $\alpha+1>\gamma\geqslant 0$ *or* (ii) $\mu>\lambda>1$, $\rho = 1/\lambda-1/\mu$, $\alpha+1>\gamma\geqslant 0$, *then*

$$|C_\alpha Q, \gamma|_\lambda \Rightarrow |C_{\alpha+\rho}Q, \gamma|_\mu.$$

Proposition (V) follows directly from the case $\alpha = 0$ of a theorem on strong Cesàro summability given by Flett (Theorem 2 in [5], where the notation $\{C,\alpha\}_k$ is used with the same meaning as $[C,\alpha+1]_k$ in the present paper). The case $\alpha > -1/k$ of this theorem is a corollary of an earlier result on strong Rieszian summability due to Glatfeld ([6, Theorem 8]; see also line 7 on p. 130 and the references there given). Proposition (VI) can be immediately derived from a result due to Flett [4, Theorem 1].

To indicate the scope of Theorems 10 and 11 we shall employ them, together with (II) and Theorem 6 (ii), to give alternative proofs of (V) (i) and (VI) (i). Parts (ii) of propositions (V) and (VI) cannot be deduced from the general theorems of the present paper; the proofs of Flett and Glatfeld, pertaining to these parts of the propositions, depend ultimately on a deep but special inequality of Hardy, Littlewood and Pólya [9] (see also [3, 120]).

Proof of (V) (i). The case $\lambda = \mu$ is a direct consequence of result (II). Suppose therefore that $\mu > \lambda$ and let $1/p = 1+1/\mu-1/\lambda$. Now $C_\rho = (h, 1/\epsilon_n^\rho)$ and

$$1/\epsilon_n^\rho = \int_0^1 t^n\phi(t)\,dt,$$

where $\phi(t) = \rho(1-t)^{\rho-1}$. Further, $\rho-1 > -1-1/\mu+1/\lambda = -1/p$ so that $p(\rho-1) > -1$. Hence $\phi(t)\in L^p(0,1)$, and the required inclusion follows by Theorem 10.

Proof of (VI) (i). Note that $C_{\alpha+\rho} = C_{\alpha+\rho}C_\alpha^{-1}C_\alpha = XC_\alpha$ where $X = (h, \epsilon_n^\alpha/\epsilon_n^{\alpha+\rho})$, and that $\epsilon_n^\alpha/\epsilon_n^{\alpha+\rho} = \int_0^1 t^n\phi(t)\,dt$, where

$$\phi(t) = \frac{\Gamma(\alpha+\rho+1)}{\Gamma(\rho)\Gamma(\alpha+1)}t^\alpha(1-t)^{\rho-1}.$$

Suppose first that $\lambda = \mu$. Then, since $\alpha-\gamma > -1$, $\rho > 0$, we see that $t^{-\gamma}\phi(t)\in L(0,1)$, and so, by Theorem 6(ii), $|C_\alpha,\gamma|_\lambda \Rightarrow |C_{\alpha+\rho},\gamma|_\lambda$. The required inclusion is an immediate consequence.

Suppose now that $\mu > \lambda$ and let $1/p = 1+1/\mu-1/\lambda$. Then, as above, $p(\rho-1) > -1$, and, since $\alpha+1-\gamma > 0$, $p(\alpha+1-\gamma-1/p) > -1$. Hence $\phi(t)\in L(0,1)$ and

$$t^{1-\gamma-1/p}\phi(t)\in L^p(0,1),$$

and the required inclusion follows by Theorem 11 (ii).

Many special inclusions can be established with the aid of the above results. As an illustration we prove the following (cf. [5, Theorem 2]):

$$[H,\alpha]_\lambda \Rightarrow [H,\beta]_\mu$$

if either $\mu\geqslant\lambda\geqslant 1$, $\beta > \alpha+1/\lambda-1/\mu$ *or* $\mu>\lambda>1$, $\beta = \alpha+1/\lambda-1/\mu$.

By (13), $C_\rho H_{\alpha-1} \simeq H_{\rho+\alpha-1}$ $(\rho > -1)$, and the result is therefore a consequence of (II) and (V). Note that α can be any real number.

6. Relations between summability processes of different types. We first prove

THEOREM 12. *If* $\lambda > 1$, $2 > \rho > -1$, *X is a Hausdorff matrix, and if* $\sum_0^\infty a_n$ *is* (i) *summable* $|C_1X, 0|_\lambda$ *and* (ii) *summable* $AC_\rho X$ *to* s, *then the series is summable* $[C_1, X]_\lambda$ *to* s.

When $\lambda = 1$ *condition* (ii) *is not required.*

Here A denotes the Abel method of summability and summability $AC_\rho X$ is to be interpreted as follows: $s_n \to s(AC_\rho X)$ means that $\sigma_n = C_\rho X(s_n) \to s(A)$, i.e. that

$$\lim_{x\to 1-} (1-x)\sum_0^\infty \sigma_n x^n = s.$$

It is known (see [1] and the references there given) that

$$C_\alpha \Rightarrow AC_\beta \Rightarrow AC_\gamma \quad (\alpha > -1, \gamma > \beta > -1). \qquad \text{(19)}$$

Proof. Let $s_n = \sum_{r=0}^n a_r$, $\tau_n = C_1X(na_n)$. Then, by hypothesis (i),

$$\frac{1}{n+1}\sum_{r=1}^n |\tau_r|^\lambda = \sum_{r=1}^n \frac{|\tau_r|^\lambda}{r} - \frac{1}{n+1}\sum_{r=1}^n (n+1-r)\frac{|\tau_r|^\lambda}{r} = o(1),$$

so that

$$na_n \to 0[C_1, C_1X]_\lambda.$$

Hence, by result (III), we have only to show that

$$s_n \to s(C_1X) \qquad \text{(20)}$$

in order to complete the proof. When $\lambda = 1$, (20) is an immediate consequence of hypothesis (i), and so hypothesis (ii) is redundant in this case.

Suppose now that $\lambda > 1$ and that $2 > \rho \geq 1 + 1/\lambda$. In view of (19) the additional restriction of ρ can be imposed without loss in generality. Let

$$C_\rho X(s_n) = w_n = \sum_{r=0}^n u_r,$$

so that, by (5),

$$nu_n = C_\rho X(na_n).$$

Then, by (ii),

$$w_n \to s(A); \qquad \text{(21)}$$

i.e. $\sum_0^\infty u_n$ is summable A to S.

Further, by result (VI), $|C_1X, 0|_\lambda \Rightarrow |C_\rho X, 0|_\mu$ $(\mu > \lambda)$ since $\rho - 1 > 1/\lambda - 1/\mu$. Hence, by (i),

$$\sum_{n=1}^\infty \frac{|nu_n|^\mu}{n} < \infty. \qquad \text{(22)}$$

Now by a Tauberian theorem of Hardy and Littlewood [8] (see also Flett [3, Theorem 4]), a consequence of (21) and (22) is that, for every $\delta > 1/\mu - 1$, $\sum_0^\infty u_n$ is summable (C, δ) to s, i.e. that

$$C_\delta(w_n) \to s. \qquad \text{(23)}$$

But μ can be taken arbitrarily large and so (23) holds for every $\delta > -1$. Consequently

$$C_{1-\rho}(w_n) = C_{1-\rho}C_\rho X(s_n) \to s$$

and, since $C_{1-\rho}C_\rho \simeq C_1$, (20) follows.

In order to establish the next theorem we require

LEMMA 4. *If Q is any matrix and either*

(i) $\lambda = \mu \geq 1, \gamma \geq 0, \alpha+1 > \gamma > \delta, \beta \geq \alpha - \gamma + \delta, \beta > -1$,

or

(ii) $\lambda > \mu \geq 1, \gamma \geq 0, \alpha+1 > \gamma > \delta, \beta > \alpha - \gamma + \delta, \beta > -1$,

then $|C_\alpha Q, \gamma|_\lambda \Rightarrow |C_\beta Q, \delta|_\mu$.

The two results incorporated in this lemma are immediate consequences of theorems due to Flett [4, Theorems 3 and 4].

THEOREM 13. *If X is a Hausdorff matrix,* $\lambda \geq 1, \alpha > \gamma > 0, \beta \geq \alpha - \gamma - 1$, *then*

$$|C_\alpha X, \gamma|_\lambda \Rightarrow [C_1, C_\beta X]_\lambda.$$

Proof. Let $Y = C_1^{-1}C_{\alpha-\gamma}X$, so that, by (11)

$$Y \simeq C_{\alpha-\gamma-1}X \quad \text{and} \quad C_{\gamma+1}Y \simeq C_\alpha X.$$

Then, by Lemma 4 and (19),

$$|C_\alpha X, \gamma|_\lambda \Rightarrow |C_\alpha X, 0|_1 \Rightarrow C_\alpha X \Rightarrow AC_\rho Y$$

for every $\rho > -1$. Further, by Lemma 4 (i),

$$|C_\alpha X, \gamma|_\lambda \Rightarrow |C_1 Y, 0|_\lambda.$$

Hence, by Theorem 12 and result (II), $|C_\alpha X, \gamma|_\lambda \Rightarrow [C_1, Y]_\lambda \Rightarrow [C_1, C_\beta X]_\lambda$.

We conclude with some corollaries of Theorems 12 and 13, but first we prove the inclusion:

$$[H, \alpha]_\lambda \Rightarrow (H, \beta) \quad (\lambda > 1, \beta > \alpha - 1 + 1/\lambda). \qquad \text{(24)}$$

By Theorem 2,

$$[H, \alpha]_\lambda = [C_1, H_{\alpha-1}]_\lambda \Rightarrow [C_{\beta-\alpha+1}, H_{\alpha-1}]_1$$

since $\beta - \alpha + 1 > 1/\lambda$. Consequently, by Theorem 3 (ii) and (13),

$$[H, \alpha]_\lambda \Rightarrow C_{\beta-\alpha+1}H_{\alpha-1} \simeq H_\beta,$$

and (24) is thus established. Alternatively, (24) can be deduced directly from the case $\mu = \infty$ of Theorem 10. By Theorem 3 (ii), the inclusion is also valid when $\lambda = 1, \beta \geq \alpha$.

Similarly we can prove the companion inclusion:

$$[C, \alpha]_\lambda \Rightarrow (C, \beta) \quad (\lambda > 1, \beta > \alpha - 1 + 1/\lambda, \alpha \geq 0).$$

This result is known (except possibly for the case $\alpha = 0$), the cases $\alpha = 1$, $\alpha > 1/\lambda$ and $\alpha > 0$ being due respectively to Kuttner [13], Hyslop [11] and Chow [2] (see also Flett [5]).

(VII). *If* $\lambda > 1$, $1 + \alpha > \rho$, *and if* $\sum_0^\infty a_n$ *is* (i) *summable* $|H, \alpha, 0|_\lambda$ *and* (ii) *summable* AH_ρ *to* s, *then the series is summable* $[H, \alpha]_\lambda$ *to* s *and consequently summable* (H, β) *to* s *for every* $\beta > \alpha - 1 + 1/\lambda$.

Proof. Let δ be a positive number such that $2 > \delta \geqslant \rho+1-\alpha$. Then, by (13), $H_\rho \Rightarrow H_\delta H_{\alpha-1} \simeq C_\delta H_{\alpha-1}$, and so, by a result due essentially to Hausdorff ([9]; see also [1, Theorem 4]),

$$AH_\rho \Rightarrow AC_\delta H_{\alpha-1}.$$

Since $H_\alpha = C_1 H_{\alpha-1}$, we obtain the required result by applying first Theorem 12 (with δ in place of ρ) and then inclusion (24).

In the same way we can prove

(VII)′. *If* $\lambda > 1$, $1+\alpha > \rho \geqslant 0$, $\beta > \alpha-1+1/\lambda$, *and if* $\sum_0^\infty a_n$ *is* (i) *summable* $|C, \alpha, 0|_\lambda$ *and* (ii) *summable* AC_ρ *to s, then the series is summable* (H, β) *to s.*

The case $\alpha = 0$, $\rho = 0$ of this result is effectively the theorem of Hardy and Littlewood used in the above proof of Theorem 12. The case $\lambda = 2$, $\rho = 0$, $\alpha > -\frac{1}{2}$, is due to Zygmund [16], and Flett [4] has established the case $\alpha > -1/\lambda$, $\rho = 0$.

(VIII). *If* $\lambda > 1$, $\gamma > 0$, $\beta > \alpha-1-\gamma+1/\lambda$, *then*

$$|H, \alpha, \gamma|_\lambda \Rightarrow [H, \alpha-\gamma]_\lambda \Rightarrow (H, \beta).$$

Proof. Let $X = C_\rho^{-1} H_\alpha$ where $\rho > \gamma$. Then $C_\rho X = H_\alpha$ and, by (13),

$$C_{\rho-\gamma-1} X \simeq H_{\alpha-\gamma-1}.$$

Consequently, by Theorem 13 and results (II) and (24),

$$|H, \alpha, \gamma|_\lambda = |C_\rho X, \gamma|_\lambda \Rightarrow [C_1, C_{\rho-\gamma-1}X]_\lambda \simeq [H_1, H_{\alpha-\gamma-1}]_\lambda = [H, \alpha-\gamma]_\lambda \Rightarrow (H, \beta).$$

A similar proof shows that

(VIII)′. *If* $\lambda > 1$, $\alpha > -1$, $\gamma > 0$, $\beta > \alpha-1-\gamma+1/\lambda$, *then*

$$|C, \alpha, \gamma|_\lambda \Rightarrow (H, \beta).$$

The case $\alpha > \gamma - 1/\lambda$ of this result has been proved by Flett [4].

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