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ON METHODS OF SUMMABILITY BASED ON INTEGRAL FUNCTIONS. II

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In two recent papers ((1), (2)) I investigated inclusion relations between IF (integral function) methods of summability. In the present paper the investigation is continued with the aid of standard Mellin transform theory and results due to Rogosinski and to Kuttner.

1. *Notation and preliminary results.* Let

$$p(z) = \sum_{n=0}^{\infty} p_n z^n$$

be an integral function such that $p_n \geqslant 0$, $\sum_{r=n}^{\infty} p_r > 0$ for all n. Associated with $p(z)$ are two regular IF methods of summability P^*, P defined as follows. We write $s_n \to l(P^*)$ i

$$\sum_{n=0}^{\infty} p_n s_n z^n = p_s(z)$$

is an analytic function at the origin having, in a region containing the positive real axis, an analytic continuation $p_s^*(z)$ such that $p_s^*(x)/p(x) \to l$ as $x \to \infty$ (through real values). And we write $s_n \to l(P)$ if $s_n \to l(P^*)$ and $p_s(z)$ is an integral function.

Let

$$\mu_n > 0, \quad q_n = p_n/\mu_n \quad (n = 0, 1, \ldots), \quad q(z) = \sum_{n=0}^{\infty} q_n z^n,$$

and, whenever $q(z)$ is an integral function, denote the associated IF methods by Q^*, Q. Following standard practice, we say that P includes Q, and write $P \supseteq Q$, if $s_n \to l(P)$ whenever $s_n \to l(Q)$ (l finite); and we say that P and Q are equivalent if each includes the other. We use the same terminology and notation for methods P^*, Q^*. In addition, we introduce for convenience the notation $P^* \dot{\supseteq} Q$ to signify that $s_n \to l(P^*)$ whenever both $s_n \to l(Q)$ and the radius of convergence of $\sum_0^{\infty} p_n s_n z^n$ is greater than zero.

We state first an easily verified lemma. As a precondition it is to be supposed that $q(z)$ is an integral function.

LEMMA 1. *Let P_ν^*, P_ν, Q_ν^*, Q_ν be the* IF *methods associated with the integral functions*

$$\sum_{n=n_0}^{\infty} p_{n+\nu} z^n, \quad \sum_{n=n_0}^{\infty} q_{n+\nu} z^n,$$

where n_0, ν are integers such that $n_0 \geqslant 0$, $n_0 + \nu \geqslant 0$. Then the relations $P \supseteq Q$, $P^* \supseteq Q^*$, $P^* \dot{\supseteq} Q$ are equivalent respectively to $P_\nu \supseteq Q_\nu$, $P_\nu^* \supseteq Q_\nu^*$, $P_\nu^* \dot{\supseteq} Q_\nu$.

The following theorem is a slight extension of a known result ((1), Theorem A).

Theorem 1. *Suppose that $\gamma > 0$, $1 \geqslant \delta > 0$, $N \geqslant 0$, $N-\nu \geqslant n_0 \geqslant 0$, where N, n_0, ν are integers, and that*

(i) *$\chi(t)$ is a function of bounded variation in the interval $[0, 1]$;*

(ii) *$\chi(t)$ is real and* $\int_0^1 t^n d\chi(t) \geqslant \delta \int_0^1 t^n |d\chi(t)| > 0 \quad (n = n_0, n_0+1, \ldots)$;

(iii) $\mu_n = \gamma^n \int_0^1 t^{n-\nu} d\chi(t) \quad (n = N, N+1, \ldots)$.

Then (iv) *$q(z)$ is an integral function;* (v) *$P \supseteq Q$, and* (vi) *$P^* \supseteq Q^*$.*

To establish this theorem we may suppose, without loss in generality, that $\gamma = 1$ and, in view of Lemma 1, that $\nu = 0$. That (iv) and (v) are consequences of the hypotheses in this case is proved in (1). That (vi) is also a consequence can be established by a similar argument incorporating the following simple result: $F(z) = \int_0^1 f(zt)\, d\chi(t)$ is an analytic function for all real $z \geqslant 0$ whenever $f(z)$ is analytic for such z and (i) holds.

Applying Lemma 1 to another known result ((2), Theorem 2), we obtain

Theorem 2. *Suppose that $1 \geqslant \delta > 0$, $\pi \geqslant \Delta > 0$, $N \geqslant 0$, $N-\nu \geqslant n_0 \geqslant 0$, where N, n_0, ν are integers, and that*

(i) *$\phi(z)$ is an analytic function in the region $|z| > 0$, $|\arg z| < \Delta$, and the integral $\int_0^\infty |\phi(t e^{i\theta})|\, dt$ is uniformly convergent (at both limits) in the interval $-\Delta < \theta < \Delta$;*

(ii) *$\phi(t)$ is real for $t > 0$ and*

$$\infty > \int_0^\infty t^n \phi(t)\, dt \geqslant \delta \int_0^\infty t^n |\phi(t)|\, dt > 0 \quad (n = n_0, n_0+1, \ldots);$$

(iii) $\mu_n = \int_0^\infty t^{n-\nu} \phi(t)\, dt \quad (n = N, N+1, \ldots)$.

Then $q(z)$ is an integral function and $P^* \doteqdot Q$.

2. *Moment sequences.* We say (as in (1)) that a sequence $\{\kappa_n\}$ is an *m-sequence* if the equations

$$\kappa_n = \int_0^1 t^n d\chi(t) \quad (n = 0, 1, \ldots)$$

admit a solution χ satisfying (i) of Theorem 1. If the equations admit a solution satisfying both (i) and (ii) of Theorem 1 we call $\{\kappa_n\}$ an *$\overline{m}$-sequence*. Further, we shall call a sequence $\{\lambda_n\}$ an *M^*-sequence* if the equations

$$\lambda_n = \int_0^\infty t^n \phi(t)\, dt \quad (n = 0, 1, \ldots)$$

have a solution ϕ satisfying (i) and (ii) of Theorem 2.

We can now reword Theorems 1 and 2 as follows:

Theorem 1′. *If $\{\kappa_n\}$ is an $\overline{m}$-sequence and, for any fixed integer ν and all large integers n, $\mu_n = \gamma^n \kappa_{n-\nu}$ $(\gamma > 0)$, then $q(z)$ is an integral function, $P \supseteq Q$, and $P^* \supseteq Q^*$.*

Theorem 2′. *If $\{\lambda_n\}$ is an M^*-sequence and, for any fixed integer ν and all large integers n, $\mu_n = \lambda_{n-\nu}$, then $q(z)$ is an integral function and $P^* \doteqdot Q$.*

We state next three useful lemmas. The first is due to Rogosinski ((7), 185–6), the second essentially to Kuttner ((6), Lemma 10), and the third sets out results given in (1).

Rogosinski's lemma. *If $f(s)$ is an analytic function of $s = \sigma + i\tau$ in the region $\sigma > c_0$, and if*

$$\int_{-\infty}^{\infty} |f(c+it)|^2 dt < K < \infty$$

for all $c > c_0$, then $$f(s) = \int_0^1 t^s \phi(t)\, dt \quad (\sigma > c_0),$$

where $t^c \phi(t) \in L(0, 1)$ for all $c > c_0$.

Kuttner's lemma. *If $A(s)$ is an analytic function of $s = \sigma + i\tau$ in the region $\sigma > \sigma_0$ such that, when $\sigma > \sigma_0$ and $|s|$ is large,*

$$A(s) = s^{\alpha s+\beta} e^{-\alpha s} \left\{C + O\left(\frac{1}{|s|}\right)\right\},$$

where C, α are positive and β is real; and if

$$F(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^{-s} A(s)\, ds \quad (c > \sigma_0),$$

then, as $t \to \infty$ (through real values)

$$F(t) \sim B\, e^{-\alpha t^{1/\alpha}}\, t^{(\beta+\frac{1}{2})/\alpha},$$

where B is a positive constant.

Lemma 2. (i) *If $\{\alpha_n\}$ and $\{\beta_n\}$ are $\overline{m}$-sequences then so also are $\{\alpha_n + \beta_n\}$ and $\{\alpha_n \beta_n\}$.*

(ii) *Any real m-sequence which converges to a positive limit is an $\overline{m}$-sequence.*

We conclude this section with two lemmas concerning $\overline{m}$- and M^*-sequences.

Lemma 3. *If*

(i) *$g(s)$ is an analytic function of $s = \sigma + i\tau$ in the region $\sigma > \sigma_0$ such that, when $\sigma > \sigma_0$ and $|s|$ is large, $g(s) = s^{-a}\{C + O(|s|^{-b})\}$, where $C > 0$, $a \geqslant 0$, $b > \frac{1}{2}$;*

(ii) *$g(\sigma)$ is real for $\sigma > \sigma_0$;*

(iii) *$\kappa_n = g(n+\nu)$ $(n = 0, 1, \ldots)$, where $\nu > 0$, $\nu - \sigma_0 > 0$;*

then $\{\kappa_n\}$ is an $\overline{m}$-sequence.

Proof. Note that

$$f(s) = g(s+\nu)\,(s+\nu)^a - C$$

satisfies the hypotheses of Rogosinski's lemma with $c_0 = \frac{1}{2}(\sigma_0 - \nu) < 0$, and that $\kappa_n(n+\nu)^a = f(n) + C \to C > 0$ as $n \to \infty$. Hence by Rogosinski's lemma and Lemma 2 (ii), $\{\kappa_n(n+\nu)^a\}$ is an $\overline{m}$-sequence. Since $\{(n+\nu)^{-a}\}$ is an $\overline{m}$-sequence, it follows, by Lemma 2 (i), that $\{\kappa_n\}$ is an $\overline{m}$-sequence.

Lemma 4. *If*

(i) *$h(s)$ is an analytic function of $s = \sigma + i\tau$ in the region $\sigma > \sigma_0$ such that, when $\sigma > \sigma_0$ and $|s|$ is large,*

$$h(s) = s^{\alpha s+\beta} e^{\gamma s} \{C + O(1/|s|)\},$$

where C, α are positive and β, γ are real;

(ii) *$h(\sigma)$ is real for $\sigma > \sigma_0$;*

(iii) *$\lambda_n = h(n+\nu)$ $(n = 0, 1, \ldots)$, where $\nu \geq 0$, $\nu - \sigma_0 > 0$;*

then $\{\lambda_n\}$ is an M^*-sequence.

Proof. Denote the principal value of $\arg s$ by ψ. Then $\psi\tau = |\psi\tau|$ and, for any $\epsilon > 0$ and $c_2 \geq \sigma \geq c_1 > \sigma_0$,

$$|\psi| > \frac{\pi}{2} - \frac{\epsilon}{2\alpha},$$

when $|\tau|$ is large. Consequently, by (i), when $|\tau| \to \infty$

$$h(s) = O\{(\sigma^2+\tau^2)^{\frac{1}{2}(\alpha\sigma+\beta)} e^{-\alpha\psi\tau}\} = O\{e^{(\epsilon-\frac{1}{2}\alpha\pi)|\tau|}\}$$

uniformly in the strip $c_2 \geq \sigma \geq c_1$. Now let $0 < \Delta < \frac{1}{2}\alpha\pi$ and let

$$\phi(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{\nu-s-1}h(s)\,ds \quad (c > \sigma_0,\ |\arg z| < \Delta).$$

In view of the above order relation, the integral is absolutely convergent and, by Cauchy's theorem, its value is independent of c. Applying a standard result on Mellin transforms ((8) Theorem 31) we obtain the following conclusions:

(i)$'$ $\phi(z)$ is analytic in the region $|z| > 0$, $|\arg z| < \Delta$;

(ii)$'$ for any $\rho > \sigma_0 - \nu$ and all $t > 0$,

$$\phi(t\,e^{i\theta}) = O(t^{-\rho-1})$$

uniformly in the interval $-\Delta < \theta < \Delta$;

(iii)$'$ for $\sigma > \sigma_0$

$$h(s) = \int_0^\infty t^{s-\nu}\phi(t)\,dt,$$

and the integral is absolutely convergent.

It follows from (i)$'$ and (ii)$'$, since $\sigma_0 - \nu < 0$, that ϕ satisfies (i) of Theorem 2. Further, by Kuttner's lemma with $A(s) = e^{-(\alpha+\gamma)s}h(s)$, it follows from (i) that when $t \to \infty$ (through real values),

$$t^{1-\nu}\phi(t\,e^{\alpha+\gamma}) \sim C'\,e^{-\alpha t^{1/\alpha}}\,t^{(\beta+\frac{1}{2})/\alpha},$$

where $C' > 0$. Hence

(iv)$'$ when $t \to \infty$, $\qquad \phi(t) \sim K\,e^{-\kappa t^{1/\alpha}}\,t^k,$

where $\qquad K > 0, \quad \kappa = \alpha\,e^{-(\gamma+\alpha)/\alpha}, \quad k = \nu - 1 + \dfrac{2\beta+1}{2\alpha}.$

Next, by the Schwartz principle of symmetry, it follows from (ii) that $\overline{h(s)} = h(\bar{s})$ and hence that

(v)$'$ $\phi(t)$ is real for $t > 0$.

In virtue of (iv)$'$ and (v)$'$, there is a number $T \geq 0$ such that $\phi(t) > 0$ for $t \geq T$. Also, by (i),

$$\lim_{n\to\infty} T^n/h(n+\nu) = 0.$$

Hence, by (iii)$'$,

$$\lim_{n\to\infty}\frac{1}{h(n+\nu)}\int_0^\infty t^n|\phi(t)|\,dt = 1 - \lim_{n\to\infty}\frac{1}{h(n+\nu)}\int_0^T t^n\{\phi(t)-|\phi(t)|\}\,dt = 1;$$

and so, for all large n,

$$h(n+\nu) = \int_0^\infty t^n\phi(t)\,dt \geq \frac{1}{2}\int_0^\infty t^n|\phi(t)|\,dt > 0.$$

It follows that ϕ satisfies (ii) of Theorem 2. Consequently, by (iii) and (iii)$'$, $\{\lambda_n\}$ is an M^*-sequence.

3. *Applications.* Suppose that $\alpha > 0$ and β is real; and denote by $(B,\alpha,\beta)^*$, (B,α,β) the IF methods associated with the integral function

$$\sum_{n=N}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},$$

where N is a non-negative integer such that $\alpha N + \beta > 0$. The actual choice of N is clearly immaterial. By Stirling's theorem,

$$\Gamma(\alpha s+\beta) = (2\pi)^{\frac{1}{2}}\,e^{-\alpha s}\,(\alpha s)^{\alpha s+\beta-\frac{1}{2}}\{1+O(1/|s|)\},$$

when $|s|$ is large and $\sigma > N$. Consequently, if $a > 0$, b is real and

$$\sigma_0 = \max(-b/a,\ -\beta/\alpha),$$

$$\frac{\Gamma(\alpha s+\beta)}{\Gamma(as+b)} = e^{\rho s}s^{(\alpha-a)s+\beta-b}\left\{C+O\left(\frac{1}{|s|}\right)\right\} \quad (C > 0,\ \rho \text{ real}),$$

when $|s|$ is large and $\sigma > \sigma_0$. Hence, by Lemma 4 and Theorem 2$'$, we obtain the following result:

(I) *if $\alpha > a > 0$ and β, b are real, then $(B,a,b)^* \doteq (B,\alpha,\beta)$.* (For a somewhat different proof of the case $\beta = b = 1$ of this result see (2).)

Further, by Lemma 3 and Theorem 1$'$, we have

(II) *if $\alpha > 0$ and $b > \beta$, then $(B,\alpha,b) \supseteq (B,\alpha,\beta)$ and $(B,\alpha,b)^* \supseteq (B,\alpha,\beta)^*$* (cf. Knopp (5)).

Suppose now that $h(s)$ satisfies (i) and (ii) of Lemma 4. Then $h(\sigma) > 0$ for $\sigma > N \geq \sigma_0$, where N is some non-negative integer; and the functions

$$\frac{h(s)\,\kappa^s}{\Gamma(\alpha s+\beta+\frac{1}{2})},\quad \frac{\Gamma(\alpha s+\beta+\frac{1}{2})}{h(s)\,\kappa^s} \quad (\kappa = \alpha^\alpha\,e^{-\alpha-\gamma})$$

satisfy the hypotheses for $g(s)$ in Lemma 3 (with $a = 0$, $b = 1$). Hence, by Lemma 3 and Theorem 1$'$, we have

(III) *if $h(s)$ satisfies* (i) *and* (ii) *of Lemma 4, then the* IF *methods associated with the integral function*

$$\sum_{n=N}^{\infty}\frac{z^n}{h(n)}$$

are equivalent respectively to $(B,\alpha,\beta+\frac{1}{2})^*$ and $(B,\alpha,\beta+\frac{1}{2})$.

An interesting special case of this result is obtained by taking

$$h(s) = \{\Gamma(as+b)\}^c\,(s+p)^{qs+r},$$

where b, c, p, q, r are real, $a > 0$ and $ac+q > 0$. This function satisfies (ii) and (i) of Lemma 4 with

$$\alpha = ac+q, \quad \beta = c(b-\tfrac{1}{2})+r, \quad \gamma = ac(\log a - 1), \quad C = e^{pq}(2\pi)^{\frac{1}{2}c}a^{(b-\frac{1}{2})c}.$$

Consequently

(IV) *the* IF *methods associated with the integral function*

$$\sum_{n=N}^{\infty} \frac{z^n}{\{\Gamma(an+b)\}^c (n+p)^{qn+r}} \quad (a > 0,\ ac+q > 0)$$

are equivalent respectively to $(B, ac+q, bc+r-\frac{1}{2}c+\frac{1}{2})^*$ *and* $(B, ac+q, bc+r-\frac{1}{2}c+\frac{1}{2})$.

In virtue of results I to IV we now have a comprehensive account of the inclusion relations between different IF methods of the above type. In connexion with these methods it is of interest to examine the behaviour of the series appearing in III and IV for large positive z. We start with the known result (see (4), 197–8)

$$v(x) = \sum_{n=0}^{\infty} \frac{x^{\alpha n}}{\Gamma(\alpha n+1)} \sim \frac{e^x}{\alpha} \quad (\alpha > 0)$$

as $x \to \infty$. Consequently, in view of a standard result of the Toeplitz type, we have, for $\delta > 0$,

$$\sum_{n=0}^{\infty} \frac{x^{\alpha n+\delta}}{\Gamma(\alpha n+\delta+1)} = \frac{e^x}{\Gamma(\delta)} \int_0^x e^{t-x}(x-t)^{\delta-1} e^{-t} v(t)\, dt \sim \frac{e^x}{\alpha}$$

as $x \to \infty$. It follows that

$$\sum_{n=N}^{\infty} \frac{x^{\alpha n+\beta-\frac{1}{2}}}{\Gamma(\alpha n+\beta+\frac{1}{2})} \sim \frac{e^x}{\alpha} \quad (\alpha > 0,\ \alpha N+\beta+\tfrac{1}{2} > 0),$$

and hence that

$$w(x) = \sum_{n=N}^{\infty} \frac{x^n}{\Gamma(\alpha n+\beta+\frac{1}{2})} \sim \frac{1}{\alpha} x^{(\frac{1}{2}-\beta)/\alpha} e^{x^{1/\alpha}}$$

as $x \to \infty$. Suppose, as above, that $h(s)$ satisfies (i) and (ii) of Lemma 4. Then, as $n \to \infty$,

$$\frac{1}{h(n)} = \frac{K\kappa^n}{\Gamma(\alpha n+\beta+\frac{1}{2})}\{1+o(1)\},$$

where $K = (2\pi)^{\frac{1}{2}}\alpha^{\beta}/C$, $\kappa = \alpha^{\alpha}e^{-\alpha-\gamma}$. Since $(B, \alpha, \beta+\frac{1}{2})$ is regular, it follows that, as $x \to \infty$,

$$\sum_{n=N}^{\infty} \frac{x^n}{h(n)} = Kw(\kappa x)\{1+o(1)\} \sim \frac{K}{\alpha}(\kappa x)^{(\frac{1}{2}-\beta)/\alpha} e^{(\kappa x)^{1/\alpha}}.$$

In particular, we find that, as $x \to \infty$,

$$\sum_{n=N}^{\infty} \frac{x^n}{\{\Gamma(an+b)\}^c (n+p)^{qn+r}} \sim \frac{K}{\alpha}(\kappa x)^{(\frac{1}{2}-\beta)/\alpha} e^{(\kappa x)^{1/\alpha}} \quad (a > 0,\ ac+q > 0),$$

where $\alpha = ac+q$, $\beta = c(b-\frac{1}{2})+r$, and $\kappa = e^{-q}\alpha^{\alpha}a^{-ac}$, $K = e^{-pq}a^{r-\beta}\alpha^{\beta}(2\pi)^{\frac{1}{2}(1-c)}$. (See Hardy (3), 55 for special cases.)

Note added in proof. Since submitting this paper for publication I have been informed that Dr L. Włodarski has proved independently the following case of result I: if $\alpha > a > 0$, $s_n \to l(B, \alpha, 1)$ and $\sum_0^{\infty} s_n z^n/\Gamma(an+1)$ is an integral function, then $s_n \to l(B, a, 1)$.

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