On the Solution of Linear Mean Recurrences

David Borwein, Jonathan M. Borwein, and Brailey Sims

Abstract. Motivated by questions of algorithm analysis, we provide several distinct approaches to determining convergence and limit values for a class of linear iterations.

1. INTRODUCTION.

Problem I. Determine the behavior of the sequence defined recursively by

$$x_n := \frac{x_{n-1} + x_{n-2} + \cdots + x_{n-m}}{m} \qquad \text{for } n \geq m+1 \tag{1}$$

and satisfying the initial conditions

$$x_k = a_k, \qquad \text{for } k = 1, 2, \ldots, m, \tag{2}$$

where $a_1, a_2, \ldots, a_m$ are given real numbers.

This problem was encountered by Bauschke, Sarada, and Wang [**1**] while examining algorithms to compute zeroes of maximal monotone operators in optimization. Questions they raised concerning its resolution motivated our ensuing consideration of various approaches whereby it might be addressed.

We suspect that, like us, the first thing most readers do when presented with a discrete iteration is to try to solve for the limit, call it L, by taking the limit in (1). Supposing the limit to exist, we deduce

$$L = \frac{\overbrace{L + L + \cdots + L}^{m}}{m} = L, \tag{3}$$

and learn nothing—at least not about the limit. There is a clue in that the result is vacuous in large part because it involves an average, or *mean*.

In the next three sections, we present three quite distinct approaches. While at least one will be familiar to many readers, we suspect that not all three will be. Each has its advantages, both as an example of more general techniques and as a doorway to a beautiful corpus of mathematics.

2. SPECTRAL SOLUTION. We start with what may well be the best-known approach. It may be found in many linear algebra courses, often along with a discussion of the Fibonacci numbers $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$, $F_1 = 1$.

Equation (1) is an example of a *linear homogeneous recurrence relation of order m* with constant coefficients. Standard theory (see, for example [**5**, Chapter 13.2, p. 252] or [**9**, Section 12.5, p. 90]) runs as follows.

http://dx.doi.org/10.4169/amer.math.monthly.121.06.486
MSC: Primary 65Q30, Secondary 65Q30

Theorem 2.1 (Linear recurrences). *The general solution of a linear recurrence*

$$x_n = \sum_{k=1}^{m} \alpha_k x_{n-k}$$

with constant coeffients, has the form

$$x_n = \sum_{k=1}^{l} q_k(n) r_k^n, \tag{4}$$

where the r_k are the l distinct roots of the characteristic polynomial

$$p(r) := r^m - \sum_{k=1}^{m} \alpha_k r^{k-1}, \tag{5}$$

with algebraic multiplicity m_k and q_k, are polynomials of degree at most $m_k - 1$.

Typically, elementary books only consider simple roots, but we shall use a little more.

Our equation analyzed. The linear recurrence relation specified by equation (1) has characteristic polynomial

$$\begin{aligned} p(r) &:= r^m - \frac{1}{m}(r^{m-1} + r^{m-2} + \cdots + r + 1) \\ &= \frac{mr^{m+1} - (m+1)r^m + 1}{m(r-1)} \end{aligned} \tag{6}$$

with roots $r_1 = 1, r_2, r_3, \ldots, r_m$. Since

$$p'(1) = m - \frac{1}{m}\sum_{n=1}^{m-1} n = m - \frac{m-1}{2} = \frac{m+1}{2},$$

the root at (1) is simple.

We next show that if $p(r) = 0$ and $r \neq 1$, then $|r| < 1$. We argue as follows. We know from (6) that $p(r) = 0$ if and only if

$$r + \frac{1}{mr^m} = 1 + \frac{1}{m}. \tag{7}$$

If $|r| > 1$, then

$$\left| r + \frac{1}{mr^m} \right| \leq |r| + \frac{1}{m|r|^m} < 1 + \frac{1}{m},$$

since the function $f(x) := x + \frac{1}{mx^m}$ is strictly increasing for real $x > 1$ and $f(1) = 1 + \frac{1}{m}$. Thus, $p(r) \neq 0$ when $|r| > 1$. Suppose therefore that $p(r) = 0$ with $r =$

$e^{i\theta}$, $0 \leq \theta < 2\pi$. Then by (7) we must have

$$\cos(\theta) + \frac{\cos(-m\theta)}{m} = 1 + \frac{1}{m},$$

which is only possible when $\theta = 0$.

By (4) we must have

$$x_n = c_1 + \sum_{k=2}^{r} q_k(n) r_k^n, \tag{8}$$

where r_k lies in the open unit disc for $2 \leq k \leq m$. Thus, the limit in (8) exists and equals $c_1 = q_k(1)$, the constant polynomial coefficient of the eigenvalue 1.

Identifying the limit. In fact, we may use (6) to see that all roots are simple. It follows from (6) that

$$((1-r)p(r))' = (m+1)r^{m-1}(1-r),$$

and hence that the only possible multiple root of p is $r_1 = 1$, which we have already shown to be simple. So the solution is actually of the form

$$x_n = c_1 + \sum_{k=2}^{m} c_k r_k^n, \quad \text{with } c_1, \ldots, c_m \text{ constants.} \tag{9}$$

Observe now that if r is any of the roots $r_2, r_3, \ldots, r_m$, then

$$\sum_{n=1}^{m} n r^n = \frac{m r^{m+2} - (m+1) r^{m+1} + r}{(r-1)^2} = \frac{m r p(r)}{r-1} = 0, \tag{10}$$

and so multiplying (9) by n and summing from $n = 1$ to m, we obtain

$$c_1 = \frac{2}{m(m+1)} \sum_{n=1}^{m} n a_n. \tag{11}$$

Thence, we do have convergence and the limit $L = c_1$ is given by (11).

Example 2.2 (The weighted mean). We may perform the same analysis, if the arithmetic average in (1) is replaced by any weighted arithmetic mean

$$W_{(\alpha)}(x_1, x_2, \ldots, x_m) := \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m$$

for strictly positive weights $\alpha_k > 0$ with $\sum_{k=1}^{m} \alpha_k = 1$. Then $W_{(1/m)} = A$ is the arithmetic mean of Problem I. As is often the case, the analysis becomes easier when we generalize. The recurrence relation in this case is

$$x_n = \alpha_m x_{n-1} + \alpha_{m-1} x_{n-2} + \cdots + \alpha_1 x_{n-m}$$

for $n \geq m+1$, with *companion matrix*

$$A_m := \begin{bmatrix} \alpha_m & \alpha_{m-1} & \cdots & \alpha_2 & \alpha_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{12}$$

The corresponding characteristic polynomial of the recurrence relation

$$p(r) := r^m - \left(\alpha_m r^{m-1} + \alpha_{m-1} r^{m-2} + \cdots + \alpha_2 r^1 + \alpha_1\right)$$

is also the characteristic polynomial of the matrix.

Clearly, $p(1) = 0$. Now suppose that r is a root of p and set $\rho := |r|$. Then the triangle inequality and the mean property of $W_{(\alpha)}$ imply that

$$\rho^m \leq \sum_{k=1}^{m} \alpha_k \rho^{k-1} \leq \max_{1\leq k\leq m} \rho^{k-1}, \tag{13}$$

and so $0 \leq \rho \leq 1$.

If $\rho = 1$ but $r \neq 1$, then $r = e^{i\theta}$ for $0 < \theta < 2\pi$ and, on observing that $r^{-m}p(r) = 0$ and equating real parts, we get

$$1 = \sum_{k=1}^{m} \alpha_k e^{i(k-m-1)\theta} = \sum_{k=1}^{m-1} \alpha_k \cos((m+1-k)\theta) + \alpha_m \cos(\theta),$$

whence $\cos(\theta) = 1$, which is a contradiction. [Alternatively, we may note that the modulus is a strictly convex function, whence $\exp(i\theta) = 1$, which is again a contradiction.] Thence, all roots other than 1 have modulus strictly less than 1.

Finally, since $p'(1) = m - \sum_{k=1}^{m}(k-1)\alpha_k \geq m - (m-1)\sum_{k=1}^{m}\alpha_k = 1$, the root at 1 is still simple. Moreover, if $\sigma_k := \alpha_1 + \alpha_2 + \cdots + \alpha_k$, then

$$p(r) = (r-1)\sum_{k=1}^{m} \sigma_k r^{k-1}. \tag{14}$$

Hence, p has no other positive real root. In particular, from (4) we again have

$$x_n = L + \sum_{k=2}^{r} q_k(n) r_k^n = L + \varepsilon_n,$$

where $\varepsilon_n \to 0$ since the root at 1 is simple while all other roots are strictly inside the unit disc—but need not be simple, as illustrated in Example 2.4.

Remark 2.3. An analysis of the proof in Example 2.2 shows that the conclusions continue to hold for nonnegative weights, as long as the highest-order term $\alpha_m > 0$.

Example 2.4 (A weighted mean with multiple roots). The polynomial

$$p(r) = r^6 - \frac{r^5 + r^4 + 16r^3 + 18r^2 + 45r + 81}{162} \tag{15}$$

$$= \frac{1}{162}(2r+1)(r-1)\left(1+9r^2\right)^2, \tag{16}$$

has a root at one and a repeated pair of conjugate roots at $\pm\frac{i}{3}$. Nonetheless, the weighted mean iteration

$$x_n = \frac{81x_{n-6} + 45x_{n-5} + 18x_{n-4} + 16x_{n-3} + x_{n-2} + x_{n-1}}{162}$$

is covered by the development of Example 2.2. The limit is

$$L := \frac{162a_6 + 161a_5 + 160a_4 + 144a_3 + 126a_2 + 81a_1}{834}. \tag{17}$$

Once found, this is easily checked (in a computer algebra system) from the Invariance principle of the next section. In fact, the coefficients were found by looking in *Maple* at the thousandth power of the corresponding matrix and converting the entries to be rational.

The polynomial was constructed by examining how to place repeated roots on the imaginary axis while preserving increasing coefficients as required in (14). One general potential form is then $p(\sigma, \tau) := (r-1)(r+\sigma)(r^2+\tau^2)^2$, and we selected $p(1/2, 1/3)$. In the same fashion,

$$p\left(\frac{1}{2}, \frac{1}{2}\right) = r^6 - \frac{16r^5 + 8r^3 + 6r^2 + r + 1}{32},$$

in which r^4 has a zero coefficient, but the corresponding iteration remains well behaved; see Remark 2.3.

We will show in Example 3.3 that the approach of the next section provides the most efficient way of identifying the limit in this generalization. (In fact, we shall discover that the numerator coefficients in (17) are the partial sums of those in (15).) Example 3.3 also provides a quick way to check the assertions about limits in the next example.

Example 2.5 (Limiting examples I). Consider first

$$A_3 := \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The corresponding iteration is $x_n = (x_{n-1} + x_{n-3})/2$ with limit $a_1/4 + a_2/4 + a_3/2$. By comparison, for

$$A_3 := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

the corresponding iteration is $x_n = (x_{n-1} + x_{n-2})/2$ with limit $(a_1 + 2a_2)/3$. This can also be deduced by considering Problem I with $m = 2$ and ignoring the third row and column. The third permutation

$$A_3 := \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

corresponding to the iteration $x_n = (x_{n-2} + x_{n-3})/2$ has limit $(a_1 + 2a_2 + 2a_3)/5$.

Finally,

$$A_3 := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has $A_3^3 = I$, and so A_3^k is periodic of period three, as is obvious from the iteration $x_n = x_{n-3}$.

We return to these matrices in Example 4.7 of the penultimate section.

3. MEAN ITERATION SOLUTION. The second approach, based on [**3**, Section 8.7], deals very efficiently with equation (1); as a bonus, the proof of convergence we give below holds for nonlinear means given positive starting values.

We say a real-valued function of M is a *strict m-variable mean* if

$$\min(x_1, x_2, \ldots, x_m) \le M(x_1, x_2, \ldots, x_m) \le \max(x_1, x_2, \ldots, x_m)$$

with equality only when all variables are equal. We observe that when M is a weighted arithmetic mean, we may take its domain to be $\mathbb{R}^m$; however, certain nonlinear means—such as $G := (x_1 x_2 \cdots x_m)^{1/m}$—are defined only for positive values of the variables.

Convergence of mean iterations. In the language of [**3**, Section 8.7], we have the following.

Theorem 3.1 (Convergence of a mean iteration). *Let M be any strict m-variable mean and consider the iteration*

$$x_n := M(x_{n-m}, x_{n-m+1}, \ldots, x_{n-1}), \tag{18}$$

so when $M = A$ we recover the iteration in (1). *Then x_n converges to a finite limit $L(x_1, x_2, \ldots, x_m)$.*

Proof. Indeed, specialization of [**3**, Section 8.7, Exercise 7] actually establishes convergence for an arbitrary strict mean; but let us make this explicit for this case.

Let $\overline{x}_n := (x_n, x_{n-1}, \ldots, x_{n-m+1})$ and let $a_n := \max \overline{x}_n$, $b_n := \min \overline{x}_n$. As noted above, for general means we need to restrict the variables to nonnegative values, but for linear means no such restriction is needed. Then for all n, the mean property implies that

$$a_{n-1} \ge a_n \ge b_n \ge b_{n-1}. \tag{19}$$

Thus, $a := \lim_n a_n$ and $b := \lim_n b_n$ exist with $a \geq b$. In particular, $\overline{x}_n$ remains bounded. Select a subsequence $\overline{x}_{n_k}$ with $\overline{x}_{n_k} \to \overline{x}$. It follows that

$$b \leq \min \overline{x} \leq \max \overline{x} \leq a \tag{20}$$

while

$$b = \min M(\overline{x}) \quad \text{and} \quad \max M(\overline{x}) = a. \tag{21}$$

Since M is a strict mean, we must have $a = b$ and the iteration converges. ■

It is both here and in Theorem 3.2 that we see the power of identifying the iteration as a mean iteration.

Determining the limit. In what follows, a mapping $L : \mathbb{D}^n \to \mathbb{R}$, where $\mathbb{D} \subseteq \mathbb{R}$, is said to be a diagonal mapping if $L(x, x, \ldots, x) = x$ for all $x \in D$.

Theorem 3.2 (Invariance principle [3]). *For any mean iteration, the limit L is necessarily a mean and is the unique diagonal mapping satisfying the Invariance principle:*

$$L(x_{n-m}, x_{n-m+1}, \ldots, x_{n-1}) = L(x_{n-m+1}, \ldots, x_{n-1}, M(x_{n-m}, x_{n-m+1}, \ldots, x_{n-1})). \tag{22}$$

Moreover, L is linear whenever M is.

Proof. We sketch the proof (details may again be found in [**3**, Section 8.7]). First, check that the limit, being a pointwise limit of means, is itself a mean and so is continuous on the diagonal.

The principle follows, since

$$L(\overline{x}_m) = \cdots = L(\overline{x}_n) = L(\overline{x}_{n+1}) = L\big(\lim_n \overline{x}_n\big) = \lim_n (x_n).$$

We leave it to the reader to show that L is linear whenever M is. ■

We note that we can mix and match arguments—if we have used the ideas of the previous section to convince ourselves that the limit exists, the Invariance principle is ready to finish the job.

Example 3.3 (A general strict linear mean). If we suppose that $M(y_1, \ldots, y_m) = \sum_{i=1}^m \alpha_i y_i$, with all $\alpha_i > 0$, and that $L(y_1, \ldots, y_m) = \sum_{i=1}^m \lambda_i y_i$ are both linear, we may solve (22) to determine that for $k = 1, 2, \ldots m - 1$ we have

$$\lambda_{k+1} = \lambda_k + \lambda_m \alpha_{k+1}. \tag{23}$$

Whence, on setting $\sigma_k := \alpha_1 + \cdots + \alpha_k$, we obtain

$$\frac{\lambda_k}{\lambda_m} = \sigma_k. \tag{24}$$

Further, since L is a linear mean, we have $1 = L(1, 1, \ldots, 1) = \Sigma_{k=1}^{m}\lambda_k$; whence, summing (3.3) from $k = 1$ to m yields $\frac{1}{\lambda_m} = \Sigma_{k=1}^{m}\sigma_k$ and so becomes

$$\lambda_k = \frac{\sigma_k}{\sum_{k=1}^{m}\sigma_k}. \tag{25}$$

In particular, setting $\alpha_k \equiv \frac{1}{m}$, we compute that $\sigma_k = \frac{k}{m}$ and so $\lambda_k = \frac{2k}{m(m+1)}$, as was already determined in (11) of the previous section.

Example 3.4 (A nonlinear mean). We may replace A by the *Hölder mean*

$$H_p(x_1, x_2, \ldots, x_m) := \left(\frac{1}{m}\sum_{i=1}^{m} x_i^p\right)^{1/p}$$

for $-\infty < p < \infty$. The limit will be $\left(\sum_{k=1}^{m}\lambda_k a_k^p\right)^{1/p}$, with λ_k as in (25). In particular, with $p = 0$ (taken as a limit), we obtain in the limit the weighted geometric mean $G(a_1, a_2, \ldots, a_m) = \prod_{k=1}^{m} a_k^{\lambda_k}$. We also apply the same considerations to weighted Hölder means.

We conclude this section with an especially neat application of the arithmetic Invariance principle to an example by Carlson [**3**, Section 8.7].

Example 3.5 (Carlson's logarithmic mean). Consider the iterations with $a_0 := a > 0$, $b_0 := b > a$, and

$$a_{n+1} = \frac{a_n + \sqrt{a_n b_n}}{2}, \qquad b_{n+1} = \frac{b_n + \sqrt{a_n b_n}}{2},$$

for $n \geq 0$. In this case, convergence is immediate since $|a_{n+1} - b_{n+1}| = |a_n - b_n|/2$.

If asked for the limit, we might make little progress. But suppose we are told that the answer is the *logarithmic mean*

$$\mathcal{L}(a, b) := \frac{a - b}{\log a - \log b},$$

for $a \neq b$ and a (the limit as $a \to b$) when $a = b > 0$. We check that

$$\mathcal{L}(a_{n+1}, b_{n+1}) = \frac{a_n - b_n}{2\log\frac{a_n + \sqrt{b_n a_n}}{b_n + \sqrt{b_n a_n}}} = \mathcal{L}(a_n, b_n),$$

since $2\log\frac{\sqrt{a_n}}{\sqrt{b_n}} = \log\frac{a_n}{b_n}$. The invariance principle of Theorem 3.2 then confirms that $\mathcal{L}(a, b)$ is the limit. In particular, for $a > 1$,

$$\mathcal{L}\left(\frac{a}{a-1}, \frac{1}{a-1}\right) = \frac{1}{\log a},$$

which quite neatly computes the logarithm (slowly) using only arithmetic operations and square roots.

4. NONNEGATIVE MATRIX SOLUTION. A third approach is to exploit directly the nonnegativity of the entries of the matrix A_m. This seems best organized as a case of the *Perron–Frobenius theorem* [**6**, Theorem 8.8.1].

Recall that a matrix A is *row stochastic* if all entries are nonnegative and each row sums to one. Moreover, A is *irreducible* if for every pair of indices i and j, there exists a natural number k such that $(A^k)_{ij}$ is not equal to zero. Recall also that the *spectral radius* is defined as $\rho(A) := \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$ [**6**, p. 177]. Since A is not assumed symmetric, we may have distinct eigenvectors for A and its transpose corresponding to the same nonzero eigenvalue. We call the latter *left eigenvectors*.

Theorem 4.1 (Perron–Frobenius, Utility grade [2, 6, 8]). *Let A be a row-stochastic irreducible square matrix. Then the spectral radius $\rho(A) = 1$ and 1 is a simple eigenvalue. Moreover, the right eigenvector $e := [1, 1, \ldots, 1_m]$ and the left eigenvector $l = [l_m, l_{m-1}, \ldots, l_1]$ are necessarily both strictly positive, and hence one-dimensional.*

In consequence,

$$\lim_{k\to\infty} A^k = \begin{bmatrix} l_m & l_{m-1} & \cdots & l_2 & l_1 \\ l_m & l_{m-1} & \cdots & l_2 & l_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ l_m & l_{m-1} & \cdots & l_2 & l_1 \\ l_m & l_{m-1} & \cdots & l_2 & l_1 \end{bmatrix}. \tag{26}$$

[We choose to consider l as a column vector with the highest-order entry at the top.]

The full version of Theorem 4.1 treats arbitrary matrices with nonnegative entries. Even in our setting, we do not know that the other eigenvalues are simple, but we may observe that this is equivalent to the matrix A being similar to a diagonal matrix D—whose entries are the eigenvalues in, say, decreasing order. Then $A^n = U^{-1}D^nU \to U^{-1}D^\infty U$, where the diagonal of $D^\infty = [1, 0, \ldots, 0_m]$. More generally, the *Jordan form* [**7**] suffices to show that (26) still follows. See [**8**] for very nice reprises of the general Perron–Frobenius theory and its multi-fold applications (and indeed [**11**]). In particular, [**8**, §4] gives Karlin's resolvent proof of Theorem 4.1.

Remark 4.2 (Collatz and Wielandt, [4, 10]). An attractive proof of Theorem 4.1, originating with Collatz and before him Perron, is to consider

$$g(x_1, x_2, \ldots, x_m) := \min_{1\le k\le m} \left\{ \frac{\sum_{j=1}^m a_{j,k}x_j}{x_k} \right\}.$$

Then the maximum, $\max_{\sum x_j=1, x_j\ge 0} g(x) = g(v) = 1$, exists and yields uniquely the Perron–Frobenius vector v (which in our case is e).

Example 4.3 (The closed form for l). The recursion we study is $\overline{x}_{n+1} = A\overline{x}_n$, where the matrix A has kth row A_k for m strict arithmetic means A_k. Hence, A is row stochastic and strictly positive and so its *Perron eigenvalue* is 1, while $A^*l = l$ shows the limit l is the left or adjoint eigenvector. Equivalently, this is also a so-called *compound iteration* $L := \bigotimes A_k$ as in [**3**, Section 8.7]; so mean arguments much as in the previous section also establish convergence. Here, we identify the eigenvector l with the corresponding linear function L since $L(x) = \langle l, x\rangle$.

Remark 4.4 (The closed form for L). Again, we can solve for the right eigenvector $l = A^* l$, either numerically (using a linear algebra package or direct iteration) or symbolically. Note that this closed form is simultaneously a generalization of Theorem 3.2 and a specialization of the general Invariance principle in [**3**, Section 8.7].

The case originating in (1) again has A being the companion matrix

$$A_m := \begin{bmatrix} a_m & a_{m-1} & \cdots & a_2 & a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

with $a_k > 0$ and $\sum_{k=1}^m a_k = 1$.

Proposition 4.5. *If for all $1 \le k \le m$ we have $a_k > 0$, then the matrix A_m^m has all entries strictly positive.*

Proof. We induct on k. Suppose that the first $k < m$ rows of A_m^k have strictly positive entries. Since

$$A_m^{k+1} = \begin{bmatrix} a_m & a_{m-1} & \cdots & a_2 & a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} A_m^k,$$

it follows that

$$(A_m^{k+1})_{1j} = \sum_{r=1}^m (A_m)_{1r} (A_m^k)_{rj} > 0,$$

and that, for $2 \le i \le k+1 \le m$,

$$(A_m^{k+1})_{ij} = \sum_{r=1}^m (A_m)_{ir} (A_m^k)_{rj} = (A_m^k)_{i-1,j} > 0.$$

Thus, the first $k+1$ rows of A_m^{k+1} have strictly positive entries, and we are done. ∎

Remark 4.6 (A picture may be worth a thousand words). The last theorem ensures the irreducibility of A_m by establishing the stronger condition that A_m^m is a strictly positive matrix.

Both the irreducibility of A_m and the stronger condition obtained above may be observed in the following alternative way. There are many equivalent conditions for the irreducibility of A. One obvious condition is that *an $m \times m$ matrix A with nonnegative entries is irreducible if (and only if) A' is irreducible, where A' is A with each of its nonzero entries replaced by 1.*

Now, A' may be interpreted as the *adjacency matrix* (see [**6**, Chapter 8]) for the directed graph G with vertices labeled $1, 2, \ldots, m$ and an edge from i to j precisely when $(A')_{ij} = 1$. In this case, the ij entry in the kth power of A' equals the number of paths of length k from i to j in G. Thus, irreducibility of A corresponds to G being strongly connected.

For our particular matrix A_m, as given in (12), the associated graph G_m is depicted in Figure 1.

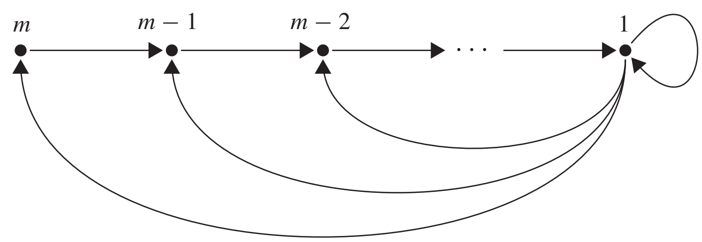

Figure 1. The graph G_m with adjacency matrix A'_m

The presence of the cycle $m \to m-1 \to m-2 \to \cdots \to 1 \to m$ shows that G_m is connected and hence that A_m is irreducible.

A moments' checking also reveals that in G_m, any vertex i is connected to any other j by a path of length m (when forming such paths, the loop at 1 may be traced as many times as necessary), thus also establishing the strict positivity of A_m^m.

Example 4.7 (Limiting examples, II). We return to the matrices of Example 2.5.

First we look again at

$$A_3 := \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then A_3^4 is coordinate-wise strictly positive (but A_3^3 is not). Thus, A_3 is irreducible despite the first row not being strictly positive. The limit eigenvector is $[1/2, 1/4, 1/4]$ and the corresponding iteration is $x_n = (x_{n-1} + x_{n-3})/2$ with limit $a_1/4 + a_2/4 + a_3/2$, where the a_i are the given initial values.

Next we consider

$$A_3 := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this case, A_3 is reducible and the limit eigenvector $[2/3, 1/3, 0]$ exists but is not strictly positive (see Remark 2.3). The corresponding iteration is $x_n = (x_{n-1} + x_{n-2})/2$ with limit $(a_1 + 2a_2)/3$. This is also deducible by considering our starting case with $m = 2$ and ignoring the third row and column.

The third case

$$A_3 := \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

corresponds to the iteration $x_n = (x_{n-2} + x_{n-3})/2$. It, like the first, is irreducible with limit $(a_1 + 2a_2 + 2a_3)/5$.

Finally,

$$A_3 := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has $A_3^3 = I$, and so A_3^k is periodic of period three—and does not converge—as is obvious from the iteration $x_n = x_{n-3}$.

5. CONCLUSION. All three approaches that we have shown have their delights and advantages. It seems fairly clear, however, that for the original problem, analysis as a mean iteration—while the least well known—is by far the most efficient and also the most elementary. Moreover, all three approaches provide for lovely examples in any linear algebra class, or any introduction to computer algebra. Indeed, they offer different flavors of algorithmics, combinatorics, analysis, algebra, and graph theory.

REFERENCES

1. H. H. Bauschke, J. Sarada, X. Wang, On moving averages (2012), available at http://arxiv.org/abs/1206.3610.
2. A. Berman, R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. SIAM, Philadelphia, 1994.
3. J. M. Borwein, P. B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*. Canadian Mathematical Society Series of Monographs and Advanced Texts, Vol. 4, Wiley, New York, 1987.
4. L. Collatz, Einschließungssatz für die charakteristischen zahlen von Matrizen, *Mathematische Zeitschrift* **48** no. 1 (1942) 221–226.
5. C.-E. Froberg, *Introduction to Numerical Analysis*. Second edition. Addison-Wesley, Reading, MA, 1969.
6. C. Godsil, G. F. Royle, *Algebraic Graph Theory*. Springer, New York, 2001.
7. G. H. Golub. C. F. van Loan, *Matrix Computations*. Third edition. Johns Hopkins University Press, Baltimore, 1996.
8. C. R. MacCluer, The many proofs and applications of Perron's Theorem, *SIAM Review*, **42** (Sep. 2000) 487–498.
9. A. M. Ostrowski, *Solution of Equations in Euclidean and Banach Spaces*. Academic Press, New York, 1973.
10. H. Wielandt, Unzerlegbare, nicht negative matrizen, *Mathematische Zeitschrift* **52** (1950) 642–648.
11. Wikipedia contributors, Perron–Frobenius theorem, *Wikipedia, The Free Encyclopedia*, available at http://en.wikipedia.org/wiki/Perron%E2%80%93Frobenius_theorem.

DAVID BORWEIN obtained two B.Sc. degrees from the University of Witwatersrand, one in engineering in 1945 and the other in mathematics in 1948. From University College, London (UK) he received a Ph.D. in 1950 and a D.Sc. in 1960. He has been at the University of Western Ontario since 1963 with an emeritus title since 1989. His main area of research has been classical analysis, particularly summability theory.
Department of Mathematics, Western University, London, ON, Canada
dborwein@uwo.ca

JONATHAN M. BORWEIN is Laureate Professor in the School of Mathematical and Physical Sciences and Director of the Priority Research Centre in Computer Assisted Research Mathematics and its Applications at the University of Newcastle. An ISI highly-cited scientist and former Chauvenet prize winner, he has published widely in various fields of mathematics. His most recent books are *Convex Functions* (with Jon Vanderwerff, Cambridge University Press, 2010) and Modern *Mathematical Computation with Maple* and *Modern Mathematical Computation with Mathematica* (with Matt Skerritt, Springer Undergraduate Mathematics and Technology, 2011 and 2012).

Centre for Computer-assisted Research Mathematics and its Applications (CARMA), School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia
jonathan.borwein@newcastle.edu.au, jborwein@gmail.com

BRAILEY SIMS is Associate Professor of mathematics in the School of Mathematical and Physical Sciences and a Principal Researcher in the Priority Research Centre in Computer Assisted Research Mathematics and its Applications at the University of Newcastle. He received a B.Sc. (Honours) in 1969 and Ph.D. in 1972 from the University of Newcastle. He lectured at the University of New England from 1972 to 1989 before returning to the University of Newcastle. His original research interests were in Banach algebras and operator theory. Later his interest switched to the geometry of Banach spaces before moving to nonlinear analysis. He is an author of over 70 research papers and is co-editor, with William A. Kirk, of the (Springer) *Handbook of Metric Fixed Point Theory*.

Centre for Computer-assisted Research Mathematics and its Applications (CARMA), School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia
brailey.sims@newcastle.edu.au