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AN EXTENSION OF A THEOREM ON THE EQUIVALENCE BETWEEN ABSOLUTE RIESZIAN AND ABSOLUTE CESÀRO SUMMABILITY

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1. Introduction. Let $\sum_{n=0}^{\infty} a_n$ be a given series and let

$$C_n^{(k)} = \binom{n+k}{n}^{-1} \sum_{r=0}^{n} \binom{n-r+k}{n-r} a_r, \quad C_k(w) = w^{-k} \sum_{n<w} (w-n)^k a_n.$$

With Flett [**4**], we say that the series is summable $| C, k, q |_p$, $k > -1$, $p \geqslant 1$, q real, if

$$\sum_{n=1}^{\infty} n^{pq+p-1} \, | \Delta C_n^{(k)} |^p < \infty,$$

where $\Delta C_n^{(k)} = C_n^{(k)} - C_{n-1}^{(k)}$. Summability $| C, k, 0 |_1$ is identical with absolute Cesàro summability (C, k), or summability $| C, k |$, as defined by Fekete [**3**].

Absolute Rieszian summability (R, k), or summability $| R, k |$, has been defined by Obreschkoff [**5, 6**] as follows : $\sum a_n$ is summable $| R, k |$, $k > 0$, if

$$\int_1^{\infty} \left| \frac{d}{du} C_k(u) \right| du < \infty.$$

It is therefore natural to say that $\sum a_n$ is summable $| R, k, q |_p$, $k > 0$, $p \geqslant 1$, if

$$\int_1^{\infty} u^{pq+p-1} \left| \frac{d}{du} C_k(u) \right|^p du < \infty.$$

For this definition to be valid it is necessary to impose the additional restriction $k > 1 - 1/p$, as can be seen from the following argument (cf. Boyd and Hyslop [**1**, 94–5]).

Let $2 \leqslant n < u < n+1$, where n is an integer such that $a_n \neq 0$. Then, for $k > 0, p \geqslant 1$,

$$\left| \frac{d}{du} C_k(u) \right| = ku^{-k-1} \left| \sum_{r=1}^{n} (u-r)^{k-1} ra_r \right|$$
$$\geqslant ku^{-k-1}(u-n)^{k-1} n \, | a_n | - ku^{-k-1} \left| \sum_{r=1}^{n-1} (u-r)^{k-1} ra_r \right|,$$

so that

$$2^p \int_n^{n+1} u^{pq+p-1} \left| \frac{d}{du} C_k(u) \right|^p du \geqslant (kn)^p \, | a_n |^p \int_n^{n+1} u^{pq-kp-1} (u-n)^{kp-p} \, du$$
$$- (2k)^p \int_n^{n+1} u^{pq-kp-1} \left| \sum_{r=1}^{n-1} (u-r)^{k-1} ra_r \right|^p du.$$

Since the final integral is finite, it follows that

$$\int_n^{n+1} u^{pq+p-1} \left| \frac{d}{du} C_k(u) \right|^p du$$

is infinite unless $kp-p > -1$, that is, unless $k > 1-1/p$.

The object of this note is to prove the following

THEOREM. *For* $p \geqslant 1$, $k > 1-1/p$, $k \geqslant q-1/p$, Σa_n *is summable* $|C, k, q|_p$ *if and only if it is summable* $|R, k, q|_p$.

The case $p = 1$, $q = 0$, of this theorem has been established by Hyslop [2]. The proof of the theorem is modelled on the one given by Boyd and Hyslop [1] for an analogous result (with $q = 0$) on strong summability. One of their subsidiary results which we use is :

LEMMA. *If* $\alpha_r \geqslant 0$, $p \geqslant 1$, $\lambda > 1-1/p$, *then*

$$\sum_{n=1}^{N}\left\{\sum_{r=1}^{n}\frac{\alpha_r}{(n+1-r)^{\lambda+1}}\right\}^p \leqslant K\sum_{n=1}^{N}\alpha_n^p,$$

where K is independent of N and α_r.

2. Proof of the theorem. Let $p \geqslant 1$, $k > 1-1/p$, $k \geqslant q-1/p$, and let n be a positive integer.

(i) It follows from an order relation given by Boyd and Hyslop [1, 97], that, for $n < u \leqslant n+1$,

$$u^{pq+p-1}\left|\frac{d}{du}C_k(u)\right|^p = O\left\{\left(u^{q-k-1/p}\sum_{r=1}^{n}\frac{r^{k+1}\,|\Delta C_r^{(k)}|}{(n+1-r)^{k+1}}\right)^p\right\}$$

$$+O\left\{(u-n)^{kp-p}\left(u^{q-k-1/p}\sum_{r=1}^{n}\frac{r^{k+1}\,|\Delta C_r^{(k)}|}{(n+1-r)^{k+1}}\right)^p\right\}$$

$$= O\left\{\left(1+(u-n)^{kp-p}\right)\left(\sum_{r=1}^{n}\frac{r^{q+1-1/p}\,|\Delta C_r^{(k)}|}{(n+1-r)^{k+1}}\right)^p\right\},$$

since $k+1/p-q \geqslant 0$; whence

$$\int_n^{n+1} u^{pq+p-1}\left|\frac{d}{du}C_k(u)\right|^p du = O\left\{\left(\sum_{r=1}^{n}\frac{r^{q+1-1/p}\,|\Delta C_r^{(k)}|}{(n+1-r)^{k+1}}\right)^p\right\},$$

since $kp-p > -1$.

It follows, by the lemma, that there is a positive number K_1 such that

$$\int_1^{\infty} u^{pq+p-1}\left|\frac{d}{du}C_k(u)\right|^p du = \sum_{n=1}^{\infty}\int_n^{n+1} u^{pq+p-1}\left|\frac{d}{du}C_k(u)\right|^p du$$

$$\leqslant K_1\sum_{n=1}^{\infty} n^{pq+p-1}\left|\Delta C_n^{(k)}\right|^p.$$

Consequently Σa_n is summable $|R, k, q|_p$ whenever it is summable $|C, k, q|_p$.

(ii) Now let m be the integer such that $m-1 \leqslant k < m$. In virtue of a result established by Boyd and Hyslop [1, 99], we find that

$$n^{q+1-1/p}\,|\Delta C_n^{(k)}| = O\left\{n^{q-k-1/p}\sum_{r=0}^{n}(n+1-r)^{k-m-2}\int_r^{r+1} u^{k+1}\left|\frac{d}{du}C_k(u)\right|du\right\}$$

$$= O\left\{\sum_{r=1}^{n}(n+1-r)^{k-m-2}\int_r^{r+1} u^{q+1-1/p}\left|\frac{d}{du}C_k(u)\right|du\right\},$$

since $k+1/p-q \geqslant 0$, and $\dfrac{d}{du}C_k(u) = 0$ for $0 < u < 1$. Applying now the lemma with

$$\alpha_r = \int_r^{r+1} u^{q+1-1/p}\left|\frac{d}{du}C_k(u)\right|du$$

and Hölder's inequality, we see that there is a positive number K_2 such that

$$\sum_{n=1}^{\infty} n^{pq+p-1}\,|\Delta C_n^{(k)}|^p \leqslant K_2\sum_{n=1}^{\infty}\left(\int_n^{n+1} u^{q+1-1/p}\left|\frac{d}{du}C_k(u)\right|du\right)^p$$

$$\leqslant K_2\sum_{n=1}^{\infty}\left(\int_n^{n+1} u^{pq+p-1}\left|\frac{d}{du}C_k(u)\right|^p du\right)\left(\int_n^{n+1} du\right)^{p-1}$$

$$= K_2\int_1^{\infty} u^{pq+p-1}\left|\frac{d}{du}C_k(u)\right|^p du.$$

Hence Σa_n is summable $|C, k, q|_p$ whenever it is summable $|R, k, q|_p$.

The proof of the theorem is thus complete.

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