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BINARY AND TERNARY TRANSFORMATIONS OF SEQUENCES

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1. Introduction

Agnew (**1**) has defined a binary transformation $T(\alpha)$, with α real, as one which takes the sequence $\{s_i\}$, $i=0, 1, \ldots$, into the sequence $\{s_i(1, \alpha)\}$ where

$$s_i(1, \alpha)=\begin{cases}\alpha s_0 & \text{for } i=0,\\ \alpha s_i+(1-\alpha)s_{i-1} & \text{for } i=1, 2, \ldots .\end{cases}$$

An r-fold application of $T(\alpha)$ yields the transformation $T^r(\alpha)$ which takes $\{s_i\}$ into $\{s_i(r, \alpha)\}$ where, in general, if $s_n(0, \alpha)=s_n$ and $s_n(r, \alpha)=0$ for negative integral n then, for all n and $l \geqslant 0$,

$$s_n(l+1, \alpha)=\alpha s_n(l, \alpha)+(1-\alpha)s_{n-1}(l, \alpha).$$

It easily follows by induction that

$$s_n(l+r, \alpha)=\sum_{k=0}^{r}\binom{r}{k}(1-\alpha)^{r-k}\alpha^k s_{n-r+k}(l, \alpha), \tag{i}$$

with the convention that $0^0=1$.

Putting $l=0$, $q=1/\alpha-1$, $(\alpha\neq 0)$, we obtain

$$s_n(r, \alpha)=(q+1)^{-r}\sum_{k=n-r}^{n}\binom{r}{n-k}q^{n-k}s_k \tag{ii}$$

and

$$s_n(n, \alpha)=(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_k. \tag{iii}$$

If $s_n(r, \alpha)$ tends to a finite limit s as n tends to infinity then $\{s_i\}$ is said to be summable $T^r(\alpha)$ to s. If $s_n(n, \alpha)$ tends to a finite limit s as n tends to infinity then $\{s_i\}$ may be said to be summable $T^\infty(\alpha)$ to s. From (iii) and Hardy (**2**), equation (8.3.4), it follows that summability $T^\infty(\alpha)$ is equivalent to Euler summability (E, q). It should also be noted that summability $T^0(\alpha)$ is equivalent to convergence.

We shall use the notation $P\Rightarrow Q$ to mean that any sequence summable (P) to s is necessarily summable (Q) to s, and $P\Leftrightarrow Q$ to mean that both $P\Rightarrow Q$ and $Q\Rightarrow P$.

2. Relations between $T^r(\alpha)$ and $T^\infty(\alpha)$

Knopp (**5**) has shown that for $0<\alpha\leqslant 1$ convergence to s implies summability $(E, 1/\alpha-1)$ to s, i.e. that $T^0(\alpha)\Rightarrow T^\infty(\alpha)$; and from a general result on compounded matrices Agnew (**1**) has deduced that $T^r(\alpha)\Rightarrow T^\infty(\alpha)$ for $r\geqslant 0$, $0<\alpha<1$.

The case $\alpha=\frac{1}{2}$ of this result was familiar to Hutton (3) who first considered the $T^r(\frac{1}{2})$ process early in the nineteenth century without giving rigorous proofs. The following proof is more direct than Agnew's.

Theorem. *For* $r\geqslant 0$,

(i) $T^r(\alpha)\Rightarrow T^{r+1}(\alpha)$ *for any* α;

(ii) $T^r(\alpha)\Rightarrow T^\infty(\alpha)$ *if and only if* $0<\alpha\leqslant 1$.

Proof. (i) is trivial.

(ii) *Sufficiency.* Let $q=1/\alpha-1\geqslant 0$ and suppose that $\{s_n\}$ is summable $T^r(\alpha)$ to s. Applying the (E, q) process, which is known (see e.g. (2), p. 179) to be regular for $q\geqslant 0$, to the sequence $s_r(r, \alpha)$, $s_{r+1}(r, \alpha)$, $s_{r+2}(r, \alpha)$, ... which converges to s, we get that

$$(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_{k+r}(r, \alpha)\to s \text{ as } n\to\infty.$$

In virtue of identity (i) with r, l and n replaced by n, r and $n+r$ respectively, it follows that $\{s_n\}$ is summable $T^\infty(\alpha)$ to s.

Necessity. If $s_n=(1-2/\alpha)^n$ then the $T^\infty(\alpha)$ transform of $\{s_n\}$ is $\{(-1)^n\}$; and for $\alpha>1$, $\{s_n\}$ is summable $T^0(\alpha)$ and so summable $T^r(\alpha)$, but is not summable $T^\infty(\alpha)$.

If $\{s_n\}$ is the sequence 1, 0, 0, ... then its $T^\infty(\alpha)$ transform is $\{(1-\alpha)^n\}$. For every α, $\{s_n\}$ is summable $T^0(\alpha)$, and so summable $T^r(\alpha)$, to 0; but the sequence is summable $T^\infty(0)$ to 1 and is not summable $T^\infty(\alpha)$ for any $\alpha<0$.

The condition $0<\alpha\leqslant 1$ is therefore necessary.

3. Nörlund means, etc.

The following results will be used later:

Kubota's theorem. (6). *If* $a_0, a_1, \ldots, a_k (a_k\neq 0)$ *are fixed real or complex numbers then, in order that* x_n *should tend to* $l/(a_0+a_1+\ldots+a_k)$ *whenever* $a_0x_{n-k}+a_1x_{n-k+1}+\ldots+a_kx_n$ *tends to* l, *it is necessary and sufficient that all roots of the equation* $a_0+a_1x+\ldots+a_kx^k=0$ *should lie within the unit circle.*

Nörlund means. Suppose that $p_0\neq 0$, $P_n=p_0+p_1+\ldots+p_n$ where p_n is real, and that $P_n\neq 0$ for $n\geqslant M$.

For $n\leqslant M$ let $t_n=\sum_{k=0}^{n}p_{n-k}s_k/P_M$

and for $n\geqslant M$ let $t_n=\sum_{k=0}^{n}p_{n-k}s_k/P_n$.

We shall say that sequence $\{s_n\}$ is summable by the Nörlund method (N, p_n) to s if t_n tends to s as n tends to infinity. In (2), Hardy imposes the further condition $p_n\geqslant 0$ (and takes $M=0$), but this is too restrictive for our purposes.

It follows from formula (ii) that for $\alpha\neq 0$ the $T^r(\alpha)$ transformation is a Nörlund transformation with

$$M=r,\quad p_n=\begin{cases}\binom{r}{n}p^n,\ p=1/\alpha-1, \text{ for } 0\leqslant n\leqslant r,\\ 0 \text{ for } n>r,\end{cases}$$

$$P_n=(1+p)^r \text{ if } n\geqslant r,$$

and $$\sum_{n=0}^{\infty}p_nx^n=(1+px)^r.$$

It is also known (2, p. 109) that the Cesàro mean (C, r) with $r\geqslant 0$ can be expressed as a Nörlund mean (N, p_n) with

$$M=0,\ p_n=\binom{n+r-1}{r-1}\sim\frac{n^{r-1}}{\Gamma(r)} \text{ if } r>0,$$

and $$p_0=1,\ p_n=0 (n=1, 2, \ldots) \text{ if } r=0.$$

For $r\geqslant 0$, $\sum_{n=0}^{\infty}p_nx^n=(1-x)^{-r}$ and $P_n\sim\dfrac{n^r}{\Gamma(r+1)}$.

The following simple extensions of Hardy's theorems 16, 17, 19 and 21 can be established by using the methods of his proofs and (in the case of theorem 17) a result due to Jurkat and Peyerimhoff (4, lemma 1).

Theorem 16. *The Nörlund method* (N, p_n) *is regular,* i.e. *the convergence of a sequence to a finite limit implies its summability* (N, p_n) *to the same limit, if and only if there is a constant* H *independent of* n *such that*

$$\sum_{r=0}^{n}|p_r|<H|P_n| \text{ for } n\geqslant M$$

and $p_n/P_n\to 0$ *as* $n\to\infty$.

Theorem 17. *Any two regular Nörlund methods* (N, p_n), (N, q_n) *are consistent*; *i.e. if a sequence is summable* (N, p_n) *to* s *and* (N, q_n) *to* t *then* $s=t$.

Theorem 19. *If* (N, p_n) *and* (N, q_n) *are regular and* $p(x)=\Sigma p_nx^n$, $q(x)=\Sigma q_nx^n$, $q(x)/p(x)=\Sigma k_nx^n$, *then in order that summability* (N, p_n) *of a sequence should imply its summability* (N, q_n) *it is necessary and sufficient that*

$$\sum_{r=0}^{n}|k_{n-r}P_r|<H|Q_n| \text{ for } n\geqslant M,$$

where H *is independent of* n, *and that* $k_n/Q_n\to 0$.

Theorem 21. *A necessary condition that two regular Nörlund methods* (N, p_n) *and* (N, q_n) *be equivalent is that* $\Sigma|k_n|$ *and* $\Sigma|l_n|$ *be finite, where* $\Sigma l_nx^n=p(x)/q(x)$.

Corollary. *Regular Nörlund methods* (N, p_n) *and* (N, q_n) *cannot be equivalent if* $p(x)$ *and* $q(x)$ *are rational and one of them has a zero, inside or on the unit circle, which is not a zero of the other.*

In the case of the $T^r(\alpha)$ process $p(x)$ has a zero at $x=-1/p=\alpha/(\alpha-1)$ if $\alpha\neq 0$; while $T^r(0)\Leftrightarrow T^r(1)$, and for $T^r(1)$ $p(x)$ has no zero. It follows from the corollary that *if $\alpha\leq\frac{1}{2}$, $\beta\leq\frac{1}{2}$ and $\alpha\neq\beta$ then $T^r(\alpha)$ and $T^s(\beta)$ cannot be equivalent for any α, β, r, s.*

4. Relation of $T^r(\alpha)$ to the Cesàro and Abel processes

If (N, p_n) is taken as the (C, s) process with $s>0$, and (N, q_n) as $T^r(\alpha)$, then

$$k(x)=(1+px)^r(1-x)^s, \quad |\,k_0\,|\,P_n\sim n^s/\Gamma(s+1)$$

and $Q_n=\alpha^{-r}$ for $n\geq r$.

By theorem 19 it follows that, *for $s>0$, summability (C, s) cannot imply summability $T^r(\alpha)$.* In the reverse direction we have the following results:

$\alpha>\frac{1}{2}$. By Kubota's theorem a sequence which is $T(\alpha)$ summable to s converges to s if and only if $|\,(\alpha-1)/\alpha\,|<1$, i.e. if and only if $\alpha>\frac{1}{2}$. Since the $T^r(\alpha)$ transform is the $T(\alpha)$ transform of the $T^{r-1}(\alpha)$ transform it follows that $(C, 0)\Leftrightarrow T^r(\alpha)$ *for* $\alpha>\frac{1}{2}$.

$\alpha=\frac{1}{2}$. Taking (N, p_n) and (N, q_n) as $T^r(\frac{1}{2})$ and (C, r) respectively we get $k(x)=(1-x^2)^{-r}$ so that

$$k_n=\begin{cases}\dbinom{n/2+r-1}{r-1} & \text{when } n \text{ is even,}\\ 0 & \text{when } n \text{ is odd.}\end{cases}$$

For large n, $k_n=O(n^{r-1})$, $k_n/Q_n=O(1/n)=o(1)$, and

$$\{\,|\,k_0\,|\,P_n+\ldots+|\,k_n\,|\,P_0\}/Q_n=O(n^r/n^r)=O(1),$$

so that $T^r(\frac{1}{2})\Rightarrow(C, r)$.

The result is "best possible" in the sense that, *for any integer r there is a sequence which is summable $T^r(\frac{1}{2})$ but which is not summable $(C, r-\delta)$ for any $\delta>0$.* This is shown by considering the example $s_n=(-1)^n n^r/\log n$, the case $r=1$ of which is due to Silverman and Szasz (7). Since $s_n\neq o(n^{r-\delta})$, $\delta>0$, the sequence $\{s_n\}$ is not summable $(C, r-\delta)$.

If, however, $s_n=(-1)^n f(n)$ where $f(n)$ is a polynomial of degree m then $s_n(1, \frac{1}{2})=(-1)^n\frac{1}{2}\{f(n)-f(n-1)\}=(-1)^n g(n)$, where $g(n)$ is a polynomial of degree $m-1$. Hence

$$s_n(r, \tfrac{1}{2})=O(n^{m-r}).$$

Putting $f(n)=n^{r+s}$ (s a non-negative integer) gives

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}(n-k)^{r+s}=O(n^s);$$

from which it easily follows, on using the identity $k=n-(n-k)$, that

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}k^s(n-k)^r=O(n^s).$$

Further, for $n-2\geq r\geq 1$, $r\geq k\geq 0$, we have

$$\begin{aligned}\frac{\log n}{\log(n-k)} &= \left\{1+\frac{\log(1-k/n)}{\log n}\right\}^{-1}\\ &= \sum_{s=0}^{r}(-1)^s\left\{\frac{\log(1-k/n)}{\log n}\right\}^s+O\{(n\log n)^{-r}\}\\ &= 1+A_1\frac{k}{n}+A_2\left(\frac{k}{n}\right)^2+\ldots+A_r\left(\frac{k}{n}\right)^r+O(n^{-r})\end{aligned}$$

where the A's are bounded functions of n independent of k. It follows that if

$$I_n=(-1)^n 2^{-r}\sum_{k=0}^{r}(-1)^k\binom{r}{k}(n-k)^r/\log(n-k)$$

then $I_n\log n=O(1)$, so that $I_n\to 0$. But I_n is $s_n(r, \frac{1}{2})$ for the sequence $\{(-1)^n n^r/\log n\}$; hence this sequence is summable $T^r(\frac{1}{2})$ to 0.

$\alpha<\frac{1}{2}$. If $\alpha=0$ then summability $T^r(\alpha)$ is trivially equivalent to convergence. Otherwise consider, as does Agnew (1), the sequence $\{s_n\}$ where $s_n=(1-1/\alpha)^n$. It is summable $T(\alpha)$ to 0 and so is also summable $T^r(\alpha)$ to 0, but $\Sigma s_n z^n$ has radius of convergence $|\,\alpha/(\alpha-1)\,|<1$ so that $\{s_n\}$ is not Abel summable. Hence *for $\alpha<\frac{1}{2}$, $\alpha\neq 0$, summability $T^r(\alpha)$ does not imply Abel summability.*

5. Ternary transformations

We may define $T(\alpha, \beta)$ to be the ternary transformation which takes $\{s_n\}$ into the sequence $\{s_n'\}$ where

$$s_0'=\alpha s_0,\ s_1'=\alpha s_1+\beta s_0 \text{ and}$$

$$s_n'=\alpha s_n+\beta s_{n-1}+(1-\alpha-\beta)s_{n-2}\ (n=2, 3, \ldots).$$

It follows immediately that $T(\alpha, 1-\alpha)$ is equivalent to $T(\alpha)$, and that the $T(\alpha, \beta)$ transformation is a Nörlund transformation (N, p_n) with $M=2$, $p_0=\alpha$, $p_1=\beta$, $p_2=1-\alpha-\beta$, $p_n=0(n\geq 3)$, $P_n=1(n\geq 2)$, $p(x)=\alpha+\beta x+(1-\alpha-\beta)x^2$.

6. Relation of $T(\alpha, \beta)$ to the $(C, 0)$ and Abel processes

Let $f(x)=\alpha x^2+\beta x+1-\alpha-\beta$, and divide the (α, β) plane into three disjoint sets as follows. Let S_1, S_2, S_2 be respectively the sets of points (α, β) for which

(1) $f(x)$ has no zeros in the region $|\,x\,|\geq 1$,

(2) $f(x)$ has at least one zero in the region $|\,x\,|>1$,

(3) $f(x)$ has two zeros, one lying on the circle $|\,x\,|=1$ and the other in the region $|\,x\,|\leq 1$.

(*a*) It is trivially evident that $T(0, 0)\Leftrightarrow(C, 0)$. Hence, by Kubota's theorem, $T(\alpha, \beta)\Leftrightarrow(C, 0)$ if and only if $(\alpha, \beta)\in S_1$.

(*b*) If $(\alpha, \beta)\in S_2$, there is a number s such that $|\,s\,|>1$ and $f(s)=0$. Hence $\Sigma s^n z^n$ has radius of convergence $|\,1/s\,|<1$, and so the sequence $\{s^n\}$ is not Abel summable. On the other hand if $s_n=s^n$ then $s_n'=s^{n-2}f(s)=0$ so that $\{s^n\}$ is summable $T(\alpha, \beta)$ to 0. Thus *summability $T(\alpha, \beta)$ does not imply summability by Abel's method for $(\alpha, \beta)\in S_2$.*

Before investigating the behaviour of $T(\alpha, \beta)$ for $(\alpha, \beta)\in S_3$, we delimit the sets S_1, S_2 and S_3. Since S_2 is the complement of $S_1 \cup S_3$ it is sufficient to consider only S_1 and S_3.

The set S_1. We show that S_1 *consists of the point* $(0, 0)$, *the part* $\beta > \frac{1}{2}$ *of the line* $\alpha = 0$ *and the region* $2\alpha+\beta>1$, $\beta<\frac{1}{2}$.

It is easily seen that $(0, \beta)\in S_1$ if and only if $\beta=0$ or $\beta>\frac{1}{2}$. It remains to prove that when $\alpha\neq 0$, $(\alpha, \beta)\in S_1$ if and only if $2\alpha+\beta>1$, $\beta<\frac{1}{2}$.

(i) Suppose that both zeros x_1, x_2 of $f(x)$ lie in $|x|<1$. Since $f(-1)\neq 0$ and $f(1)=1$, $f(-1)=1-2\beta$ must be positive, for otherwise $f(x)$ would have one real zero in the range $-1<x<1$ and another outside this range. Hence $\beta<\frac{1}{2}$.

Further, $-1<x_1x_2=-1+(1-\beta)/\alpha<1$, so that $0<(1-\beta)/\alpha<2$. Since $\beta<\frac{1}{2}$, α must be positive and so $2\alpha+\beta>1$.

(ii) Suppose $2\alpha+\beta>1$, $\beta<\frac{1}{2}$. Then $\alpha>0$ and, as above, $-1<x_1x_2<1$. Hence, if the zeros of $f(x)$ are not real, both must lie in $|x|<1$. If both zeros are real, one must lie in the range $-1<x<1$ and, since $f(-1)>0$, $f(1)>0$, so must the other.

The set S_3.

(i) $(\alpha, \beta)\in S_3$ and $f(x)$ has non-real zeros x_1 and x_2 if and only if

$$x_1x_2=1=-1+(1-\beta)/\alpha,\ 4\alpha>(\beta+2\alpha)^2,$$

which is equivalent to $2\alpha+\beta=1$, $\alpha>\frac{1}{4}$.

(ii) Since $f(1)=1$, $(\alpha, \beta)\in S_3$ and $f(x)$ has real zeros if and only if

$$f(-1)=1-2\beta=0,\ |(1-\beta)/\alpha-1|\leqslant 1,$$

which is equivalent to $\beta=\frac{1}{2}$, $\alpha\geqslant\frac{1}{4}$.

Hence S_3 *consists of the part* $\alpha>\frac{1}{4}$ *of the line* $2\alpha+\beta=1$ *and the part* $\alpha\geqslant\frac{1}{4}$ *of the line* $\beta=\frac{1}{2}$.

7. Relation of $T(\alpha, \beta)$ to the Cesàro process in S_3

(i) *The segment* $\alpha\geqslant\frac{1}{4}$ *of the line* $\beta=\frac{1}{2}$.

Here

$$\alpha+\beta x+(1-\alpha-\beta)x^2=\alpha(1+x)\left(1+\frac{1-2\alpha}{2\alpha}x\right).$$

In theorem 19 take (N, p_n) to be the $T(\alpha, \beta)$ process and (N, q_n) the Cesàro (C, s) process. Then

$$k(x)=1/\left\{\alpha(1-x)^{s-1}(1-x^2)\left(1-\frac{2\alpha-1}{2\alpha}x\right)\right\}$$

$$=\frac{1}{\alpha}\{1+(s-1)x+\ldots\}(1+x^2+x^4+\ldots)\left\{1+\frac{2\alpha-1}{2\alpha}x+\left(\frac{2\alpha-1}{2\alpha}x\right)^2+\ldots\right\}.$$

If $s=1$, then

$$k_n=\frac{1}{\alpha}\left\{\left(\frac{2\alpha-1}{2\alpha}\right)^n+\left(\frac{2\alpha-1}{2\alpha}\right)^{n-2}+\ldots+\left(\frac{2\alpha-1}{2\alpha}\right)^{1\text{ or }0}\right\}.$$

Hence $k_n=O(1)$ if $\alpha>\frac{1}{4}$ and $|k_n|\sim 2n$ if $\alpha=\frac{1}{4}$. Also, $Q_n=n+1$. All the conditions of theorem 19 are satisfied if $\alpha>\frac{1}{4}$. Hence *for* $\alpha>\frac{1}{4}$, $T(\alpha, \frac{1}{2})\Rightarrow(C, 1)$, *but there exists a sequence summable* $T(\frac{1}{4}, \frac{1}{2})$ *which is not summable* $(C, 1)$.

If $s=2$ and $\alpha=\frac{1}{4}$ then $k(x)=4(1-x^2)^{-2}$ and the conditions of theorem 19 are easily seen to hold, so that $T(\frac{1}{4}, \frac{1}{2})\Rightarrow(C, 2)$.

(ii) *The segment* $\alpha>\frac{1}{4}$ *of the line* $2\alpha+\beta=1$.

With (N, p_n) and (N, q_n) as the $T(\alpha, \beta)$ and (C, s) processes respectively,

$$k(x)=1/\left\{\alpha(1-x)^s\left(1+\frac{1-2\alpha}{\alpha}x+x^2\right)\right\}$$

$$=\frac{1}{\alpha}(1-x)^{-s}(1-\gamma x)^{-1}\left(1-\frac{x}{\gamma}\right)^{-1}$$

$$=(1-x)^{-s}\sum_{n=0}^{\infty}(a\gamma^n+b\gamma^{-n})x^n$$

where $\gamma=\{2\alpha-1+i\sqrt{(4\alpha-1)}\}/2\alpha$ and a, b are constants. If $s=1$, then

$$k_n=\sum_{r=0}^{n}(a\gamma^r+b\gamma^{-r})=O(1)$$

since $\gamma\neq 1$ or -1 and $|\gamma|=1$. It follows that $T(\alpha, \beta)\Rightarrow(C, 1)$ *when* $\alpha>\frac{1}{4}$, $2\alpha+\beta=1$.

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