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ON METHODS OF SUMMABILITY BASED ON INTEGRAL FUNCTIONS

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1. *Introduction.* Suppose throughout that

$$p_n \geqslant 0, \quad \sum_{r=n}^{\infty} p_r > 0 \quad (n = 0, 1, \ldots)$$

and that
$$p(x) = \sum_{n=0}^{\infty} p_n x^n$$

is an integral function. Suppose also that l, s_n $(n = 0, 1, \ldots)$ are arbitrary complex numbers and denote by $\rho(ps)$ the radius of convergence of the series

$$\sum_{n=0}^{\infty} p_n s_n x^n.$$

If $\rho(ps) = \rho > 0$ and there is a function $p_s^*(z)$ such that

$$p_s^*(x) = \sum_{n=0}^{\infty} p_n s_n x^n \quad (0 \leqslant x < \rho),$$

$p_s^*(x)$ is analytic for all positive x,

$p_s^*(x)/p(x) \to l$ when $x \to \infty$ (through real values),

we write
$$s_n \to l\,(P^*).$$

If $\rho(ps) = \infty$ and $s_n \to l\,(P^*)$, we write

$$s_n \to l\,(P).$$

This defines the IF (integral function) methods of summability P^*, P. It is known ((5), p. 80) that P is regular, i.e. $s_n \to l\,(P)$ whenever $s_n \to l$; consequently P^* is also regular.

Suppose in what follows that

$$\mu_n > 0, \quad q_n = p_n/\mu_n \quad (n = 0, 1, \ldots).$$

Let $\rho(q)$ be the radius of convergence of

$$\sum_{n=0}^{\infty} q_n x^n = q(x),$$

and, whenever $\rho(q) = \infty$, denote the IF methods associated with the sequence $\{q_n\}$ by Q^*, Q. Suppose also that N is an arbitrary non-negative integer.

The following theorem has been proved elsewhere (Borwein (1)).

Theorem A. *If*

$$\mu_n = \int_0^1 t^n d\chi(t) \geqslant \delta \int_0^1 t^n |d\chi(t)| > 0 \quad (\delta > 0, n \geqslant N)$$

where $\chi(t)$ is a real function of bounded variation in $[0,1]$, then $\rho(q) = \infty$ and $s_n \to l(P)$ whenever $s_n \to l(Q)$.

In this paper two further theorems of the same type are proved and are used in conjunction with Theorem A to obtain inclusion relations between some special IF methods of summability. The first of these theorems is

Theorem 1. *If $\chi(t)$ is a real function of bounded variation in $(0,\infty)$ such that*

$$\infty > \int_0^\infty t^n d\chi(t) \geqslant \delta \int_0^\infty t^n |d\chi(t)| > 0 \quad (\delta > 0, n \geqslant N)$$

and if

$$\mu_n = \int_0^\infty t^n d\chi(t) \quad (n \geqslant N),$$

then $\rho(q) = \infty$.

If, in addition, $s_n \to l(Q)$ and $\rho(ps) = \infty$, then $s_n \to l(P)$.

Before formulating the second theorem we define a class Ω of functions $\phi(z)$ as follows:

$\phi \in \Omega$ if there are positive numbers Δ, δ and a non-negative integer N such that

(Ω_1) $\phi(z)$ is analytic in the region $|z| > 0$, $-\Delta < \arg z < \Delta$; and, when $\tau \to 0+$, $T \to \infty$, the integrals

$$\int_\tau^1 |\phi(t\,e^{i\theta})|\,dt, \quad \int_1^T |\phi(t\,e^{i\theta})|\,dt$$

tend to finite limits uniformly in the interval $-\Delta < \theta < \Delta$;

(Ω_2) $\phi(t)$ is real for $t > 0$ and

$$\infty > \int_0^\infty t^n \phi(t)\,dt \geqslant \delta \int_0^\infty t^n |\phi(t)|\,dt > 0 \quad (n \geqslant N).$$

Theorem 2. *If $\phi \in \Omega$ and*

$$\mu_n = \int_0^\infty t^n \phi(t)\,dt \quad (n \geqslant N),$$

then $\rho(q) = \infty$.

If, in addition, $s_n \to l(Q)$ and $\rho(ps) > 0$, then $s_n \to l(P^*)$.

2. *Proofs of Theorems* 1 *and* 2. Suppose in what follows that

$$p_n = q_n = 0 \quad (0 \leqslant n \leqslant N);$$

it is evident that this leads to no real loss in generality in either theorem.

Let $\chi(t)$, μ_n satisfy the hypotheses of Theorem 1. Then, since

$$\infty > \int_0^\infty t^{N+1} |d\chi(t)| > 0,$$

there is a number $a > 0$ such that

$$\infty > K = \delta \int_a^\infty |d\chi(t)| > 0,$$

and so

$$p_n \geqslant \delta q_n \int_a^\infty t^n |d\chi(t)| \geqslant K a^n q_n \geqslant 0. \tag{1}$$

Suppose now that

$$q_s(x) = \sum_{n=0}^\infty q_n s_n x^n$$

is an integral function and that $\rho(ps) = \rho > 0$. Then

$$\sum_{n=0}^\infty p_n s_n x^n = \sum_{n=0}^\infty s_n x^n q_n \int_0^\infty t^n d\chi(t) = \int_0^\infty q_s(xt)\,d\chi(t) \quad (0 \leqslant x < \rho); \tag{2}$$

the inversion being legitimate since, for $0 \leqslant x < \rho$,

$$\infty > \delta^{-1} \sum_{n=0}^\infty p_n |s_n| x^n \geqslant \sum_{n=0}^\infty |s_n| x^n q_n \int_0^\infty t^n |d\chi(t)| = \int_0^\infty |d\chi(t)| \sum_{n=0}^\infty q_n |s_n| (xt)^n.$$

Further, taking $s_n = 1$, we get

$$p(x) = \int_0^\infty q(xt)\,d\chi(t) \geqslant \delta \int_0^\infty q(xt)\,|d\chi(t)| \quad (x \geqslant 0). \tag{3}$$

Proof of Theorem 1. Since $p(x)$ is an integral function, it follows from (1) that $\rho(q) = \infty$. We now suppose that $s_n \to l(Q)$ and that $\rho(ps) = \rho = \infty$. In view of (3) we have, for $x > w > 0$,

$$\left|\frac{1}{p(x)} \int_0^\infty q_s(xt)\,d\chi(t) - l\right|$$

$$\leqslant \frac{1}{p(x)} \left| \int_0^{w/x} \{q_s(xt) - lq(xt)\}\,d\chi(t) \right| + \frac{1}{p(x)} \left| \int_{w/x}^\infty \left\{\frac{q_s(xt)}{q(xt)} - l\right\} q(xt)\,d\chi(t) \right|$$

$$\leqslant \frac{1}{p(x)} \int_0^\infty |d\chi(t)| \sum_{n=0}^\infty q_n(|s_n| + |l|)\,w^n + \delta^{-1} \operatorname*{bd}_{v \geqslant w} \left|\frac{q_s(v)}{q(v)} - l\right|.$$

Since $p(x) \to \infty$ and $q_s(x)/q(x) \to l$ when $x \to \infty$, it follows that

$$\lim_{x\to\infty} \frac{1}{p(x)} \int_0^\infty q_s(xt)\,d\chi(t) = l,$$

and hence, by (2), that $s_n \to l(P)$.

Proof of Theorem 2. In view of conditions (Ω_2) on ϕ, the hypotheses of Theorem 1 are satisfied by

$$\chi(t) = \int_0^t \phi(u)\,du \quad (t \geqslant 0),$$

and consequently, as above, $\rho(q) = \infty$. Suppose that $s_n \to l(Q)$ and that $\rho(ps) = \rho > 0$. Then, by (2),

$$\sum_{n=0}^\infty p_n s_n x^n = \int_0^\infty q_s(xt)\,\phi(t)\,dt \quad (0 \leqslant x < \rho).$$

Since ϕ satisfies conditions (Ω_1) and $q_s(t)$ is continuous and bounded for $t > 0$, it is easily seen that, when $\tau \to 0+$, $T \to \infty$, the integrals

$$\int_\tau^1 q_s(t)\,\phi\left(\frac{t}{z}\right) dt, \quad \int_1^T q_s(t)\,\phi\left(\frac{t}{z}\right) dt$$

tend to finite limits uniformly in the region $-\Delta < \arg z < \Delta$, $C > |z| > 0$, where C is any positive number. Hence, by standard results ((6), §§ 2·84, 2·85), the function

$$p_s^*(z) = \frac{1}{z}\int_0^\infty q_s(t)\,\phi\left(\frac{t}{z}\right)dt$$

is analytic in the region $|z| > 0$, $-\Delta < \arg z < \Delta$. Also, for $x > 0$,

$$p_s^*(x) = \int_0^\infty q_s(xt)\,\phi(t)\,dt,$$

and so, as in the proof of Theorem 1,

$$\lim_{x\to\infty} p_s^*(x)/p(x) = l.$$

It follows that $s_n \to l\,(P^*)$.

3. *Lemmas.* We first show that

$$\phi(z) = z^\alpha e^{-\gamma z^\beta} \quad (\alpha > -1, \beta > 0, \gamma > 0)$$

is in the class Ω, it being assumed that for real λ, $z^\lambda = |z|^\lambda e^{i\lambda\theta}$ where θ is the principal value of $\arg z$.

It is evident that ϕ satisfies conditions (Ω_2). Also, $\phi(z)$ is analytic in the region $|z| > 0$, $|\arg z| < \pi/3\beta = \Delta$; and, for $t > 0$, $-\Delta < \theta < \Delta$,

$$|\phi(t\,e^{i\theta})| = t^\alpha e^{-\gamma t^\beta \cos\beta\theta} < t^\alpha e^{-\frac{1}{2}\gamma t^\beta} = M(t)$$

say. Since $\int_0^\infty M(t)\,dt < \infty$, we see that ϕ also satisfies conditions (Ω_1); and so $\phi \in \Omega$.

Next we prove three lemmas.

LEMMA 1. *If* $\alpha_r > 0$, $\gamma_r > \beta_r > 0$ $(r = 1, 2, \ldots, k)$, *then*

$$\mu_n = \prod_{r=1}^k \frac{\Gamma(\alpha_r n + \beta_r)}{\Gamma(\alpha_r n + \gamma_r)} \quad (n = 0, 1, \ldots)$$

satisfies the conditions of Theorem A.

Proof. Let $\alpha > 0$, $\gamma > \beta > 0$; then

$$\lambda_n = \frac{\Gamma(\alpha n + \beta)}{\Gamma(\alpha n + \gamma)} = \frac{1}{\alpha\Gamma(\gamma - \beta)}\int_0^1 t^n t^{(\beta-\alpha)/\alpha}(1 - t^{1/\alpha})^{\gamma-\beta-1}dt$$

is totally monotone. It follows ((5), §§ 11·8, 11·9) that μ_n is totally monotone and hence that it satisfies the conditions of Theorem A.

LEMMA 2. *If* $\alpha > 0$, $\beta > 0$, $1 > \gamma > 0$, *then*

$$\mu_n = \frac{\Gamma(\alpha n + \beta)}{\Gamma(\gamma\alpha n + \gamma\beta)} \quad (n = 0, 1, \ldots)$$

satisfies the conditions of Theorem 1.

Proof. It has been shown by Good (3) that

$$\mu_n = \int_0^\infty t^n \phi(t)\,dt$$

where
$$\alpha\pi\phi(t) = \sum_{n=0}^\infty (-1)^{n+1}\frac{\Gamma(\gamma n + 1)\sin\gamma n\pi}{n!}t^{(n+\beta-\alpha)/\alpha},$$

the series being convergent for all $t > 0$. When $t \to 0+$,

$$\alpha\pi\phi(t) \sim \Gamma(\gamma + 1)\sin\gamma\pi\, t^{(1+\beta-\alpha)/\alpha},$$

and Good has proved ((3), p. 150) that, when $t \to \infty$,

$$\phi(t) \sim K t^a e^{-ct^b}$$

where
$$K = \{2\pi(1-\gamma)\,\alpha^{1/(1-\gamma)}\}^{-\frac{1}{2}}, \quad a = \frac{1}{2\alpha(1-\gamma)} + \frac{\beta}{\alpha} - 1,$$

$$b = \frac{1}{\alpha(1-\gamma)}, \quad c = (1-\gamma)\,\gamma^{\gamma/(1-\gamma)}.$$

It follows that $\int_0^\infty t^n|\phi(t)|\,dt < \infty$ $(n = 0, 1, \ldots)$, and that there is a number $T \geq 0$ such that $\phi(t) > 0$ for $t \geq T$. Consequently

$$\mu_n = \int_0^\infty t^n\,|\phi(t)|\,dt - \int_0^T t^n\{|\phi(t)| - \phi(t)\}\,dt,$$

and hence, since $T^n/\mu_n \to 0$,

$$\lim_{n\to\infty}\frac{1}{\mu_n}\int_0^\infty t^n|\phi(t)|dt = 1.$$

Therefore
$$\infty > \int_0^\infty t^n\phi(t)\,dt \geq \tfrac{1}{2}\int_0^\infty t^n|\phi(t)|\,dt > 0$$

for all n sufficiently large; and so μ_n satisfies the conditions of Theorem 1 with

$$\chi(t) = \int_0^t \phi(u)\,du.$$

LEMMA 3. *If* α, β, γ, κ *are positive, then*

$$\mu_n = \kappa\gamma^{-\alpha n}\Gamma(\alpha n + \beta) \quad (n = 0, 1, \ldots)$$

satisfies the conditions of Theorem 2.

Proof. In order to establish this lemma we have only to observe that

$$\alpha\mu_n = \kappa\gamma^\beta\int_0^\infty t^n t^{(\beta-\alpha)/\alpha} e^{-\gamma t^{1/\alpha}}\,dt,$$

and that
$$\phi(z) = z^{(\beta-\alpha)/\alpha} e^{-\gamma z^{1/\alpha}}$$
is in the class Ω.

4. *Special inclusions.* (i) Let

$$p_n^\alpha = 1/\Gamma(\alpha n + 1) \quad (\alpha > 0, n = 0, 1, \ldots),$$

and denote the IF methods of summability associated with the sequence $\{p_n^\alpha\}$ by P_α^*, P_α. P_1 is then the Borel exponential method.

In a recent paper (7) Włodarski stated the following result:

If $\alpha = 2^{-k}$, $\beta = 2^{-k-h}$ $(k = 0, 1, \ldots; h = 1, 2, \ldots)$, *and if the series*

$$\sum_{n=0}^{\infty} s_n t^{\alpha n}/\Gamma(\alpha n+1) = w_\alpha(t), \quad \sum_{n=0}^{\infty} s_n t^{\beta n}/\Gamma(\beta n+1) = w_\beta(t)$$

are convergent for all $t > 0$ *and*

$$\lim_{t\to\infty} \alpha\, e^{-t} w_\alpha(t) = l,$$

then

$$\lim_{t\to\infty} \beta\, e^{-t} w_\beta(t) = l.$$

Now it is known ((5), pp. 197–8) that, for $\delta > 0$,

$$\lim_{x\to\infty} \delta\, e^{-x^{1/\delta}} \sum_{n=0}^{\infty} x^n/\Gamma(\delta n+1) = 1,$$

and so Włodarski's result is equivalent to:

If $\alpha = 2^{-k}, \beta = 2^{-k-h}$ $(k = 0, 1, \ldots; h = 1, 2, \ldots)$, *and if* $s_n \to l\,(P_\alpha)$ *and* $\Sigma s_n x^n/\Gamma(\beta n+1)$ *is an integral function, then* $s_n \to l\,(P_\beta)$.

We shall prove the more general result:

I. *If* $\alpha > \beta > 0$, *and if* $s_n \to l\,(P_\alpha)$ *and the radius of convergence of* $\Sigma s_n x^n/\Gamma(\beta n+1)$ *is greater than zero, then* $s_n \to l\,(P^*_\beta)$.

Proof. Let a, b be integers such that

$$a > b \geqslant 2, \quad b > \beta, \quad \frac{\alpha}{\beta} > \frac{a}{b},$$

and let

$$\gamma = \frac{a\beta}{b\alpha}, \quad c = a-b, \quad m = \frac{n\beta+1}{b} \quad (n = 0, 1, \ldots).$$

Then, using Gauss's multiplication theorem ((2), p. 225) for gamma functions, we get

$$\begin{aligned}\frac{p_n^\beta}{p_n^\alpha} &= \Gamma(cm+c)\,\frac{\Gamma(\alpha n+1)}{\Gamma(\gamma\alpha n+a/b)}\,\frac{\Gamma(am)}{\Gamma(bm)\,\Gamma(cm+c)}\\ &= \kappa\lambda^{-cm}\Gamma(cm+c)\,\frac{\Gamma(\alpha n+1)}{\Gamma(\gamma\alpha n+\gamma)}\,\frac{\Gamma(\gamma\alpha n+\gamma)}{\Gamma(\gamma\alpha n+a/b)}\\ &\qquad\qquad \times \prod_{r=1}^{b-1}\frac{\Gamma(m+r/a)}{\Gamma(m+r/b)}\prod_{s=b}^{a-1}\frac{\Gamma(m+s/a)}{\Gamma\{m+1+(s-b)/c\}}\\ &= \kappa\lambda^{-cm}\Gamma(cm+c)\,\frac{\Gamma(\alpha n+1)}{\Gamma(\gamma\alpha n+\gamma)}\,\mu_n,\end{aligned}$$

where $\kappa = (2\pi a)^{-\frac{1}{2}} b^{\frac{1}{2}} c^{\frac{1}{2}-c}$, $\lambda = c b^{b/c} a^{-a/c}$, and, by Lemma 1, μ_n satisfies the conditions of Theorem A.

Now let

$$p_n = \mu_n p_n^\alpha, \quad q_n = \frac{\Gamma(\alpha n+1)}{\Gamma(\gamma\alpha n+\gamma)}\,p_n,$$

so that

$$p_n^\beta = \kappa\lambda^{-cm}\Gamma(cm+c)\,q_n.$$

Suppose that

$$s_n \to l\,(P_\alpha),$$

and that the radius of convergence of $\Sigma s_n p_n^\beta x^n$ is $\rho > 0$. Then, since $\lim\,(q_n/p_n^\beta)^{1/n} = 0$, $\Sigma s_n q_n x^n$ is an integral function. Also, by Theorem A,

$$s_n \to l\,(P).$$

Consequently, by Lemma 2 and Theorem 1,

$$s_n \to l\,(Q),$$

and hence, by Lemma 3 and Theorem 2,

$$s_n \to l\,(P^*_\beta).$$

Remark. The above proof can be basically simplified if we impose the additional condition that α/β be rational. For then we can avoid the use of Lemma 2, and consequently of Good's asymptotic approximation, by putting $a/b = \alpha/\beta$ and $\gamma = 1$.

(ii) For $\alpha > 0$, denote by Q^*_α, Q_α the IF methods of summability associated with the sequence $\{(n\,!)^{-\alpha}\}$. It is known that, when $x \to \infty$,

$$\sum_{n=0}^{\infty} (n!)^{-\alpha} x^n \sim (2\pi)^{\frac{1}{2}(1-\alpha)} \alpha^{-\frac{1}{2}} x^{-\frac{1}{2}(1-1/\alpha)}\, e^{\alpha x^{1/\alpha}}$$

(see Hardy (4), p. 55).

We prove:

II. *If* $\alpha > \beta > 0$ *and* $\alpha - \beta$ *is an integer, and if* $s_n \to l\,(Q_\alpha)$ *and the radius of convergence of* $\Sigma s_n (n!)^{-\beta} x^n$ *is greater than zero, then* $s_n \to l\,(Q^*_\beta)$.

Proof. Let $k = \alpha - \beta$. Then

$$\frac{(n!)^{-\beta}}{(n!)^{-\alpha}} = \Gamma(kn+k)\,\frac{\{\Gamma(n+1)\}^k}{\Gamma(kn+k)} = \mu_n \mu_n',$$

where

$$\mu_n = \prod_{r=0}^{k-1} \frac{\Gamma(n+1)}{\Gamma(n+1+r/k)},$$

$$\mu_n' = (2\pi)^{\frac{1}{2}(1-k)} k^{k-\frac{1}{2}} k^{kn}\,\Gamma(kn+k).$$

The proof can now be completed (as above in (i)) by first appealing to Lemma 1 and Theorem A and then to Lemma 3 and Theorem 2.

We conclude by considering, in relation to result II, the sequences $\{s_n\}$, $\{t_n\}$, where

$$s_n = (-1)^n\, n!\, a^n, \quad t_n = (-1)^n (n!)^2 a^n \quad (a > 0).$$

Following Hardy ((5), p. 80) we find that $s_n \to 0\,(Q_2)$, but, since $\Sigma s_n x^n/n\,!$ is divergent when $x > 1/a$, the Borel method Q_1 cannot be applied to the sequence $\{s_n\}$. However,

$$e^{-x} \sum_{n=0}^{\infty} s_n x^n/n! = e^{-x}/(1+ax) \quad (0 \leqslant x < 1/a),$$

and so, as expected in view of II, $s_n \to 0\,(Q^*_1)$.

Next, it is easily verified that

$$t_n \to 0\,(Q_3), \quad t_n \to 0\,(Q^*_2).$$

On the other hand, $\Sigma t_n x^n/n!$ has zero radius of convergence and so the method Q^*_1 cannot be applied to $\{t_n\}$.

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