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THE NON-LOCAL NATURE OF THE SUMMABILITY OF FOURIER SERIES BY CERTAIN ABSOLUTE RIESZ METHODS[1]

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ABSTRACT. For a large class of sequences $\{\lambda_n\}$ the summability at a point of a Fourier series $\sum A_n(t)$ by the absolute Riesz method $|R, \lambda_n, 1|$ is not a local property of the generating function. On the other hand, for every $\epsilon > 0$, the $|R, \lambda_n, 1|$ summability of the factored series $\sum A_n(t)\lambda_n^{-\epsilon}$ at any point is always a local property of the generating function.

Suppose throughout that, for $n = 1, 2, \ldots$,

$$\mu_n > 0, \qquad \lambda_n := \mu_1 + \mu_2 + \cdots + \mu_n \to \infty,$$

and $s_n := a_1 + a_2 + \cdots + a_n$. The series $\sum a_n$ is said to be summable by the absolute Riesz method $|R, \lambda_n, 1|$ if

$$c(w) := \frac{1}{w} \sum_{\lambda_n < w} (w - \lambda_n) a_n$$

is of bounded variation over (λ_1, ∞), and it is said to be summable by the absolute weighted mean method $|M, \mu_n|$ if the sequence of means $\{t_n\}$ defined by

$$t_n := \frac{1}{\lambda_n} \sum_{\nu=1}^{n} \mu_\nu s_\nu$$

is of bounded variation, that is if

$$\sum_{n=1}^{\infty} |\Delta t_n| < \infty,$$

where $\Delta t_n := t_n - t_{n+1}$. It is well-known, and easily verified, that these two methods are equivalent.

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Let

$$\tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} A_n(t) := \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty}(\alpha_n \cos nt + \beta_n \sin nt)$$

be the Fourier series generated by a periodic function F with period 2π which is Lebesgue integrable over $(-\pi, \pi)$. It is familiar that the convergence of the Fourier series at $t = x$ is a local property of F (i.e. depends only on the behaviour of F in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of F. On the other hand, Bosanquet and Kestleman [5] showed that the summability $|C,1|\,(= |M,1|)$ of the Fourier series at any point is not a local property of F, and Mohanty [8] subsequently showed that this is also the case with summability $|R, \lambda_n, 1|$ when $\lambda_n := \ell_k(n)$ for n sufficiently large, where

$$\ell_0(x) := x \quad \text{and} \quad \ell_k(x) := \log(\ell_{k-1}(x))$$

for $k = 1, 2, \ldots$ and x sufficiently large. Mohanty also showed that the $|R, \log n, 1|$ summability of the factored Fourier series

$$\sum_{n=2}^{\infty} A_n(t)/\log n$$

at any point is a local property of F, whereas the $|C, 1|$ summability of this series is not. Matsumoto [6] improved the first of these results by showing that the $|R, \log n, 1|$ summability of the series

$$\sum_{n=3}^{\infty} A_n(t)(\log\log n)^{-p},\ p > 1,$$

at any point is a local property of F, and Bhatt[3] went a step further by showing that the factor $(\log\log n)^{-p}$ in the above series can be replaced by the more general factor $\gamma_n \log n$ where $\{\gamma_n\}$ is a convex sequence such that $\sum \gamma_n/n$ is convergent. Mishra [7] proved that if $\{\gamma_n\}$ is as above, and if

$$\lambda_n = O(n\mu_n) \quad \text{and} \quad \lambda_n \Delta\mu_n = O(\mu_n \mu_{n+1}),$$

then the summability $|M, \mu_n|$ of the series

$$\sum_{n=1}^{\infty} A_n(t)\gamma_n \frac{\lambda_n}{n\mu_n}$$

at any point is a local property of F. This does not directly generalize any of the above mentioned results involving $|R, \log n, 1|$ summability since the order relations are not satisfied by $\mu_n := 1/n$. Bor [4] recently showed that $|M, \mu_n|$ in Mishra's result can be replaced by a more general summability method $|M, \mu_n|_k$.

The following two theorems include most of the above mentioned results as special cases. Their proofs will appear in the Proceedings of the American Mathematical Society.

THEOREM 1. *Suppose that a is a positive integer, and that f is a positive, unbounded function with an absolutely continuous positive derivative on $[e^a, \infty)$ such that, on this interval,*

$$\frac{xf'(x)}{f(x)} \text{ decreases to } 0$$

and

$$xf''(x) = O(f'(x)).$$

Suppose also that

$$\lambda_n := f(e^n) \text{ for } n \geqq a,$$

and that $0 < \alpha < \beta < 2\pi$. Then there is a function F, Lebesgue integrable over (α, β) and zero in the remainder of $(0, 2\pi)$, whose Fourier series is not summable $|R, \lambda_n, 1|$ at $t = 0$.

This shows that, subject to the hypotheses of the theorem, the summability $|R, \lambda_n, 1|$ of a Fourier series at any point is not a local property of its generating function. Since the hypotheses are satisfied by $f(x) := \ell_k(x)$ for $k = 1, 2, \ldots$, Bosanquet and Kestleman's result, and also Mohanty's result, on the non-local nature of the summability of a Fourier series by certain absolute methods are special cases of Theorem 1.

THEOREM 2. *Suppose that the sequence $\{c_n\}$ is such that*

(1) $$\sum_{n=1}^{\infty} \frac{\mu_n}{\lambda_n} |c_n| < \infty$$

and

(2) $$\sum_{n=1}^{\infty} |\Delta c_n| < \infty.$$

Then the summability $|R, \lambda_n, 1|$ of the factored Fourier series

$$\sum_{n=1}^{\infty} A_n(t) c_n$$

at any point is a local property of the generating function F.

This theorem is a special case of a result due to Baron [2, Theorem 3]. The proof that is to appear in the Proceedings of the American Mathematical Society is, however, somewhat simpler and more direct than Baron's, partly because he deals with more general summability methods. Baron has established many results concerning local properties of Fourier series in [1] and [2]. Theorem 2 generalizes Bhatt's above-mentioned result, since it is known (see [3] for references) that if $\{\gamma_n\}$ is a convex sequence such that $\sum \gamma_n/n$ is convergent, then

$$\gamma_n \geqq \gamma_{n+1} \geqq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \log n \, \Delta\gamma_n < \infty,$$

and so (1) and (2) are satisfied by $\mu_n := 1/n$, $c_n := \gamma_n \log n$. Since, by Dini's theorem, $\sum \mu_n \lambda_n^{-1-\epsilon}$ is convergent whenever $\epsilon > 0$, we have the following corollary of Theorem 2.

COROLLARY. *For $\epsilon > 0$, the summability $|R, \lambda_n, 1|$ of the factored Fourier series*

$$\sum_{n=1}^{\infty} A_n(t)\lambda_n^{-\epsilon}$$

at any point is a local property of the generating function F.

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