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Square lattice Ising model susceptibility: connection matrices and singular behaviour of $\chi^{(3)}$ and $\chi^{(4)}$

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Abstract

We present a simple, but efficient, way to calculate connection matrices between sets of independent local solutions, defined at two neighbouring singular points, of Fuchsian differential equations of quite large orders, such as those found for the third and fourth contribution ($\chi^{(3)}$ and $\chi^{(4)}$) to the magnetic susceptibility of the square lattice Ising model. We deduce all the critical behaviours of the solutions $\chi^{(3)}$ and $\chi^{(4)}$, as well as the asymptotic behaviour of the coefficients in the corresponding series expansions. We confirm that the newly found quadratic singularities of the Fuchsian ODE associated with $\chi^{(3)}$ are not singularities of the particular solution $\chi^{(3)}$ itself. We use the previous connection matrices to get the exact expressions of all the monodromy matrices of the Fuchsian differential equation for $\chi^{(3)}$ (and $\chi^{(4)}$) expressed in the same basis of solutions. These monodromy matrices are the generators of the differential Galois group of the Fuchsian differential equations for $\chi^{(3)}$ (and $\chi^{(4)}$), whose analysis is just sketched here. As far as the physics implications of the solutions are concerned, we find challenging qualitative differences when comparing the corrections to scaling for the full susceptibility χ at high temperature (respectively low temperature) and the first two terms $\chi^{(1)}$ and $\chi^{(3)}$ (respectively $\chi^{(2)}$ and $\chi^{(4)}$).

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1. Introduction

Since a pioneering, and quite monumental, paper [1] on the two-dimensional Ising models, it has been known that the magnetic susceptibility of square lattice Ising model can be written

[1] as an infinite sum of $(n-1)$-dimensional integrals [2–7] contributions:

$$\chi(T) = \sum_{n=1}^{\infty} \chi^{(n)}(T). \tag{1}$$

The odd (respectively even) n correspond to the high (respectively low) temperature domain. These $(n-1)$-dimensional integrals are known to be holonomic, since they are integrals of holonomic (actually algebraic) integrands. In addition to the known $\chi^{(1)}$ and $\chi^{(2)}$ terms, which can be expressed in terms of simple algebraic or hypergeometric functions, it is only recently that the Fuchsian differential equations satisfied by the $\chi^{(3)}$ and $\chi^{(4)}$ terms have been found [8–10]. These two exact differential equations of quite large orders (7 and 10) can be used to find answers to a set of problems traditionally known to be subtle, and difficult, for functions with confluent singularities, such as the fine-tuning of the singular behaviours for all the singularities (dominant singular behaviour, subdominant, etc), accurate calculations of the asymptotic behaviour of the coefficients, etc.

Recall that the third and fourth contribution to the magnetic susceptibility, $\chi^{(3)}$ and $\chi^{(4)}$, are given by multi-integrals and each is, thus, a particular solution of the corresponding differential equation. These differential equations exhibit a finite set of regular singular points that may (or may not) appear in the physical solutions $\chi^{(3)}$ and $\chi^{(4)}$. In addition to the physical singularities and the non-physical singularities $s = \pm i$ (where $s = \sinh(2K)$, K being the usual Ising model coupling constant, $K = \beta J$), it is commonly believed that the $\chi^{(n)}$ have, at least, other non-physical singularities given by Nickel [6, 7]. The dominant singular behaviours at all these (non-physical) singularities ($\chi^{(3)}$ and $\chi^{(4)}$) have also been given by Nickel. The differential equations of the $\chi^{(n)}$, which 'encode' all the information on the solutions and their singular behaviour, in fact, allow us to obtain not only the dominant, but also all the subdominant singular behaviours, hardly detectable from straight series analysis. It is thus of interest to get (or confirm) these singular behaviours from the exact Fuchsian differential equations that we have actually obtained for $\chi^{(3)}$ and $\chi^{(4)}$ and, especially, the singular behaviour at the two new quadratic singularities, $1 + 3w + 4w^2 = 0$ (where $w = s/(1+s^2)/2$) found for $\chi^{(3)}$ [8].

The physical solution $\chi^{(3)}$ is defined by a double integral on two angles and is known as a series obtained by expansion (then integration) of the double integral at $w = 0$ (or $s = 0$). It is certainly not simple to obtain the $\chi^{(3)}$ expansion around (say) the ferromagnetic critical point $w = 1/4$, due to a singular logarithmic behaviour. However, one can overcome this difficulty since, with a differential equation, it is straightforward to obtain the formal series solutions at each regular singular point (i.e., a local basis of series solutions). By connecting the formal solutions around $w = 0$ and the formal series solutions around another regular singular point like $w = 1/4$, one will be able to express the particular solution $\chi^{(3)}$ (and also all the other formal solutions) as a linear combination of solutions valid at $w = 1/4$. The seven local solutions at $w = 0$ will, then, be given by the product of a 7×7 matrix with the vector having the seven local solutions at $w = 1/4$ as entries. In other words, succeeding in obtaining these connection matrices amounts to building a common (global) basis of solutions valid for all the regular singular points. Furthermore, with these connection matrices, we obtain, in fact, the analytic continuation in the whole complex plane of the variable w, of $\chi^{(3)}$ and $\chi^{(4)}$, which are known as integral representations.

Note that, remarkably, the Fuchsian differential equation for $\chi^{(3)}$ has simple rational and algebraic solutions. These rational or algebraic solutions, known in the closed form, can be understood *globally*. One can easily expand such *globally defined* solutions around any singular point of the ODE, and follow these solutions through any 'jump' from one regular singularity to another and, therefore, from one well-suited basis to another well-suited basis. For a function not known in the closed form, like the 'physical' solution $\chi^{(3)}$, the

decomposition on each well-suited local basis associated with every singular point of the ODE, is far from clear. The correspondence between these various (well-suited) local bases associated with each singular point of the ODE, is typically a *global* problem and, thus, a quite difficult one. One clearly needs to build *effective methods* to find such connection matrices in the case of Fuchsian differential equations of order 7, or 10 ($\chi^{(3)}$ and $\chi^{(4)}$), or of much higher orders ($\chi^{(5)}$, $\chi^{(6)}$, etc). With a method of matching of series, we will show that the connection matrices matching these various well-suited bases of series solutions can be obtained explicitly. The entries of these matrices can be calculated with *as many digits as we want*. We will show that we can actually find the exact expressions of these entries as simple algebraic expressions of (in the case of the Fuchsian ODEs of $\chi^{(3)}$ and $\chi^{(4)}$) powers of π, $\ln(2)$, $\ln(3)$ and various algebraic numbers or integers, together with more 'transcendental' numbers like the 'ferromagnetic constant' I_3^+ introduced in equation (7.12) of [1]:

$$I_3^+ = \frac{1}{2\pi^2}\int_1^\infty\int_1^\infty\int_1^\infty dy_1\,dy_2\,dy_3\left(\frac{y_2^2-1}{(y_1^2-1)(y_3^2-1)}\right)^{1/2} Y^2$$
$$= 0.000\,814\,462\,565\,662\,504\,439\,391\,217\,128\,562\,721\,9978\ldots \qquad (2)$$
$$Y = \frac{y_1-y_3}{(y_1+y_2)(y_2+y_3)(y_1+y_2+y_3)}$$

Focusing on $\chi^{(3)}$, and since this physical solution is known as a series expansion at $w = 0$ (low- or high-temperature expansions), we will give all the connection matrices between this $w = 0$ regular singular point and all the other regular singularities of the differential equation including the two new complex regular singularities [8, 9] which are roots of $1 + 3w + 4w^2 = 0$. We will comment on the occurrence of the 'ferromagnetic constant' I_3^+ in the various blocks of the connection matrices. The decomposition of $\chi^{(3)}$ in the well-suited basis for each regular singular point allows us to find all the singular behaviours of the physical solution. From these results, we will deduce the asymptotic behaviour of the coefficients of the series expansion of $\chi^{(3)}$. These last problems are interesting, *per se*, for series expansions analysis of lattice statistical mechanics, since they correspond to subtle analysis of confluent singularities. Actually, we will see that even the last asymptotic evaluation problem is a (global) connection problem since the physical solution like $\chi^{(3)}$ does not correspond to the obvious dominant singular behaviour one might have imagined from the indicial equation.

Focusing on the two new singularities, the roots of $1 + 3w + 4w^2 = 0$, we will show that the physical solution $\chi^{(3)}$ *is not singular at these points*. The factor of the logarithmic term, in the decomposition of $\chi^{(3)}$ at these singular points, is known exactly and vanishes identically.

Note that a fundamental concept to understand (the symmetries, the solutions of) these exact Fuchsian differential equations is the so-called *differential Galois group* [11]. Differential Galois groups have been calculated for simple enough second-order, or even third-order, ODEs (see, for instance, [12]). However, finding the differential Galois group of such higher order Fuchsian differential equations (order 7 for $\chi^{(3)}$, order 10 for $\chi^{(4)}$) with eight regular singular points (for $\chi^{(3)}$) is not an easy task [12] and requires the computation of all the monodromy matrices associated with each (non-apparent) regular singular point, considered *in the same basis*[4].

We will give the exact expression of all the monodromy matrices expressed in the same ($w = 0$) basis of solutions, these eight matrices being the generators of the differential Galois group, which will be given in a forthcoming publication [14].

[4] These monodromy matrices are the generators of the monodromy group which identifies with the differential Galois group when there are no irregular singularities, and thus no Stokes matrices [13].

This method can be generalized, *mutatis mutandis*, to the Fuchsian differential equation of $\chi^{(4)}$. Here, we give the connection matrix between $w = 0$ and both the ferromagnetic and anti-ferromagnetic critical points. The singular behaviour is straightforwardly obtained with the asymptotic behaviour of the series coefficients of the physical solution $\chi^{(4)}$. The monodromy matrices, expressed in the *same basis* of solutions are also obtained.

The paper is organized as follows. We recall, in section 2, some results on the Fuchsian differential equation satisfied by $\chi^{(3)}$, and give a *new factorization* for the corresponding order 7 differential operator, yielding the emergence of an order 2, and an order 3, differential operator (denoted by Z_2 and Y_3 below). We give, in section 3, the connection matrices matching the (series) solutions around the regular singular point $w = 0$ and around all the other regular singular points. With these connection matrices we deduce the singularity behaviour and the asymptotics on the physical solution of this ODE (section 4). In section 5, we deduce the exact expressions of the monodromy matrices expressed *in the same basis*. Section 6 generalizes these results to the Fuchsian differential equation satisfied by $\chi^{(4)}$. Some physics implications of our results at scaling are discussed in section 7. Our conclusion is given in section 8.

2. The order 7 operator L_7

Let us first recall, with the same notation as in [8, 9], the seven linearly independent solutions given in [8, 9] for the order 7 differential operator L_7, associated with[5] $\tilde{\chi}^{(3)}$.

One finds two remarkable *rational and algebraic* solutions of the order 7 differential equation associated with $\tilde{\chi}^{(3)}$, namely,

$$\mathcal{S}(L_1) = \frac{w}{1-4w}, \qquad \mathcal{S}(N_1) = \frac{w^2}{(1-4w)\sqrt{1-16w^2}} \tag{3}$$

associated with the two order 1 differential operators given in [8]:

$$L_1 = \frac{\mathrm{d}}{\mathrm{d}w} - \frac{1}{w(1-4w)}, \qquad N_1 = \frac{\mathrm{d}}{\mathrm{d}w} - \frac{2(1+2w)}{w(1-16w^2)}. \tag{4}$$

There is a solution behaving like w^3, that we denote by S_3,

$$S_3 = w^3 + 3w^4 + 22w^5 + 74w^6 + 417w^7 + 1465w^8 + 7479w^9 + 268\,39w^{10} + \cdots \tag{5}$$

and three solutions with logarithmic terms given by equation (17) in [8]. Note the singled-out series expansion starting with w^9, corresponding to the physical solution $\tilde{\chi}^{(3)}$:

$$S_9 = \frac{\tilde{\chi}^{(3)}(w)}{8} = w^9 + 36w^{11} + 4w^{12} + 884w^{13} + 196w^{14} + \cdots. \tag{6}$$

The choice of this set of linearly independent solutions (and of these series) is, in fact, arbitrary since any linear combination of solutions is also a solution of the differential equation. Three of the above solutions are, however, singled out: the solutions $\mathcal{S}(L_1)$ and $\mathcal{S}(N_1)$ which are *global* (since they have closed expression), and the series S_9 associated with the highest critical exponent in the indicial equation $(w^9 + \cdots)$, which has a unique (well-defined) expression and happens to correspond to the 'physical' solution $\tilde{\chi}^{(3)}$. Linear combinations, like $S_3 - \alpha \cdot S_9$, are, at first sight, on the same footing.

Nevertheless, introducing such a specific linear combination, Nickel[6] has been able to show that the resulting series for the particular value $\alpha = 16$ is, also, the solution of a linear

[5] $\tilde{\chi}^{(n)}$ is defined as $\chi^{(n)} = (1-s^4)^{1/4}/s \cdot \tilde{\chi}^{(n)}$, for n odd.

[6] We thank B Nickel for kindly communicating this result.

Table 1. Critical exponents for each regular singular point for the differential operators $Z_2 \cdot N_1$ and $Y_3 \cdot Z_2 \cdot N_1$. The columns N show the number of solutions with logarithmic terms. The columns P show the maximum power of the logarithm occurring in the solutions.

w-singularity	$Z_2 \cdot N_1$	N	P	$Y_3 \cdot Z_2 \cdot N_1$	N	P
0	2, 1, 1	1	1	3, 2, 2, 1, 1, 1	3	2
$-1/4$	1, 0, $-1/2$	0	0	2, 1, 0, 0, 0, $-1/2$	2	2
$1/4$	-1, -1, $-3/2$	1	1	0, 0, 0, -1, -1, $-3/2$	3	2
∞	1, 0, 0	1	1	2, 1, 1, 1, 0, 0	3	2
$-1/2$	3, 1, 0	1	1	4, 3, 3, 2, 1, 0	1	1
1	3, 1, 0	1	1	4, 3, 3, 2, 1, 0	1	1
$\frac{-3\pm i\sqrt{7}}{8}$	1, 1, 0	1	1	4, 3, 2, 1, 1, 0	1	1

differential equation of lower order, namely order 4. With this result, the factorization scheme of L_7 becomes[7]

$$\begin{aligned} L_7 &= M_1 \cdot Y_3 \cdot Z_2 \cdot N_1 = B_3 \cdot X_1 \cdot Z_2 \cdot N_1 \\ &= B_3 \cdot B_2 \cdot O_1 \cdot N_1 = B_3 \cdot B_2 \cdot T_1 \cdot L_1 \end{aligned} \tag{7}$$

where the indices correspond to the order of the differential operators (B_3, Y_3 are order 3, B_2, Z_2 order 2, and so on). The differential operators L_7, M_1 and T_1 have been given in [8]. We give in appendix A, the differential operators X_1, Z_2 and Y_3. With these differential operators, all the factorizations (7) can be found by left and right division.

From these factorizations of L_7, one can see that the general solution of the corresponding differential equation is the *direct* sum of the solution of L_1 and of the general solution of the differential operator $L_6 = Y_3 \cdot Z_2 \cdot N_1$. The operator L_7 has the following decomposition:

$$L_7 = L_6 \oplus L_1. \tag{8}$$

We thus consider, from now on, the differential operator L_6.

The formal solutions of L_6 (at the singular point $w = 0$) show the occurrence of three Frobenius series and three solutions carrying logarithmic terms. With the factorizations (7), it is interesting to see which operator brings with it a singular behaviour for a given regular singular point. Table 1 shows the critical exponents at each regular singular point for both differential operators $Z_2 \cdot N_1$ and $Y_3 \cdot Z_2 \cdot N_1$. In the third and sixth column, the number of independent solutions with logarithmic terms is shown.

At the singular points $w = 1$, $w = -1/2$, and at the two roots w_1, w_2 of $1 + 3w + 4w^2 = 0$, we remark that the solution carrying a logarithmic term is in fact a solution of $Z_2 \cdot N_1$. Therefore, the three solutions of the differential operator $Y_3 \cdot Z_2 \cdot N_1$, emerging from Y_3, are analytical at the non-physical singular points $w = 1$, $w = -1/2$, and at the quadratic roots of $1 + 3w + 4w^2 = 0$. At the singular point $w = 1/4$, we also note that the differential operator $Z_2 \cdot N_1$ is responsible for the $(1 - 4w)^{-1}$ behaviour. We will then expect the 'ferromagnetic constant' I_3^+ to be localized in the blocks of the connection matrix corresponding to the solutions of the order 3 differential operator $Z_2 \cdot N_1$ at the point $w = 1/4$.

As far as explicit calculations are concerned, a well-suited basis necessary for explicitly writing connection matrices exists and can be described. Considering the order 6 operator $L_6 = Y_3 \cdot Z_2 \cdot N_1$, we construct the local solutions, sequentially, as the global solution of N_1 then the two solutions coming from $Z_2 \cdot N_1$, to which we add the three further solutions coming from $Y_3 \cdot Z_2 \cdot N_1$. We will use below this well-suited basis.

[7] The order 4 differential operator found by B Nickel corresponds to $B_2 \cdot T_1 \cdot L_1 = B_2 \cdot O_1 \cdot N_1 = X_1 \cdot Z_2 \cdot N_1$.

3. Connection matrices for $\tilde{\chi}^{(3)}$

Using a very simple method, let us show, in the case where one has an exact Fuchsian differential equation, that one can actually very simply, and very efficiently, obtain the connection matrices between two sets of series solutions valid at two different points. The method consists in equating, at some matching points, the two sets of series corresponding, respectively, to expansions around $w = 0$ and, for instance, $w = 1/4$. The matching point should be in the radius of convergence of both series. The singular points (i.e., $w = 0$ and $w = 1/4$) should be neighbours, having no other singularity in between. Recall that the differential equation for $\tilde{\chi}^{(3)}$ has eight regular singular points, the point at infinity, five on the real axis and two (w_1 and w_2) on the upper and lower half-plane each. At a given singular point w_s, the solutions are obtained as series in the variable x, where $x = w$ (respectively $x = 1/w$) for the point $w_s = 0$ (respectively $w_s = \infty$) and $x = 1 - w/w_s$ for the other regular singular points. We take the definition $\ln(x) = \ln(-x) + \mathrm{i}\pi$ for negative values of x which corresponds to matching points in the lower (respectively upper) half-plane for $w > 0$ (respectively $w < 0$).

The computation of the connection matrix should be more efficient when two 'neighbouring' singularities are, as far as possible, far away from the other singularities and especially when the test points chosen half-way are, as far as possible, far from the other singularities, in order not to be 'polluted' by the other singularities. We remark that one can calculate in this way just 'neighbouring' singularities: connection matrices of two singularities w_1, w_r that are not 'neighbours' should be deduced using some path of 'neighbouring' connection matrices:

$$C(w_1, w_r) = C(w_1, w_2) \cdot C(w_2, w_3) \cdots C(w_{r-1}, w_r). \tag{9}$$

This is the prescription we take for the singular points on the real axis and the singularity w_1 lying in the upper half-plane. For the singularity w_2 lying in the lower half-plane, the connection matrix is calculated from

$$C(0, w_2) = C^*(0, -1/4) \cdot C^*(-1/4, w_1) = C^*(0, w_1) \tag{10}$$

where * denotes the complex conjugate.

Let us remark that changing the variable w we are working with, to the more traditional $s = \sinh(2K)$ variable, or the usual high-temperature (respectively low-temperature) variable $t = \tanh(K)$, or the variable $\tau = (1/s - s)/2$, modifies the distribution of singularities in the complex plane and their radii of convergence. However, the method can still be used. One can use that freedom in the choice of the expansion variable to actually improve the convergence of our calculations.

3.1. Connecting solutions

Let us first show, as an example, how we compute the connection matrix between two neighbouring regular singular points ($w = 0$ and $w = 1/4$) for order 3 differential operator $Z_2 \cdot N_1$. Around the singular point $w = 0$, the local solutions are two Frobenius series (one being the global solution $\mathcal{S}(N_1)$) and a series with a logarithmic term. The chosen basis is then (where $x = w$)

$$S_1^{(0)}(x) = \mathcal{S}(N_1)(x), \qquad S_2^{(0)}(x) = [0, 1, 5, 26, 106, 484, \ldots], \tag{11}$$

$$S_3^{(0)}(x) = S_2^{(0)}(x) \cdot \ln(x) + S_{30}^{(0)}(x) \tag{12}$$

with

$$S_{30}^{(0)}(x) = [0, 0, 0, 6, 26, 529/3, 2149/3, \ldots] \tag{13}$$

where $[a_0, a_1, a_2, \ldots]$ denotes the series $a_0 + a_1 x + a_2 x^2 + \cdots$. There are three independent series $S_1^{(0)}$, $S_2^{(0)}$ and $S_{30}^{(0)}$, since the operator $Z_2 \cdot N_1$ is of order 3. Similarly, around $w = 1/4$, the local solutions read (with $x = 1 - 4w$ and, where again, $S_1^{(1/4)}$ is the global solution corresponding to operator N_1)

$$S_1^{(1/4)}(x) = \mathcal{S}(N_1)(x), \tag{14}$$

$$S_2^{(1/4)}(x) = \frac{1}{x} - \frac{3}{4} - \frac{5}{96} \cdot x - \frac{3}{64} \cdot x^2 - \frac{1801}{55\,296} \cdot x^3 + \cdots \tag{15}$$

$$S_3^{(1/4)}(x) = S_2^{(1/4)}(x) \cdot \ln(x) + S_{30}^{(1/4)}(x) \tag{16}$$

with

$$S_{30}^{(1/4)}(x) = [3/8, -367/5760, -193/6720, -244\,483/6635\,520, \ldots]. \tag{17}$$

The series $S_i^{(0)}$ are defined around $w = 0$, and are convergent in a radius of $1/4$, which corresponds to the nearest regular singular point (i.e., $w = 1/4$). Similarly, the solutions $S_i^{(1/4)}$ are convergent in the disc centred at $w = 1/4$ with the same radius (i.e., $1/4$). Between the points $w = 0$ and $w = 1/4$, there is a region where both sets of solutions ($S_i^{(0)}$ and $S_i^{(1/4)}$) are convergent. This region corresponds to the common area between two discs centred, respectively, at $w = 0$ and $w = 1/4$, with the same radius $1/4$.

Connecting the local series solutions at the regular singular points $w = 0$ and $w = 1/4$ amounts to finding the 3×3 matrix $C(0, 1/4)$ such that

$$S^{(0)} = C(0, 1/4) \cdot S^{(1/4)} \tag{18}$$

where $S^{(0)}$ (respectively $S^{(1/4)}$) denotes the vector with entries $S_i^{(0)}$ (respectively $S_i^{(1/4)}$). The solutions $S_i^{(0)}$ and $S_i^{(1/4)}$ are evaluated at three arbitrary points around a point x_c belonging to both convergence discs of the series solutions $S_i^{(0)}$ and $S_i^{(1/4)}$.

Equation (18) is thus a linear system of nine unknowns. The entries of the connection matrix $C(0, 1/4)$ are obtained in the floating point form with a large number of digits. These entries are 'recognized' in the symbolic form and matrix $C(0, 1/4)$ then reads

$$C(0, 1/4) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -\dfrac{9\sqrt{3}}{64\pi}\left(\dfrac{2}{3} - \ln(24)\right) & -\dfrac{9\sqrt{3}}{64\pi} \\ 0 & -\dfrac{3\pi\sqrt{3}}{32} & 0 \end{bmatrix}. \tag{19}$$

The entries of this matrix are combinations of radicals, of powers of π and logarithms of integers. Note that there is no straightforward manner to recognize numerical values such as those displayed above. However, it is possible, in a 'tricky' way, to get rid of the logarithms of integers in the entries, and obtain as many zero entries as possible. This is shown, in the following, for this very example.

The series, in the set of local solutions $S_i^{(1/4)}$, are solutions of the differential equation (ODE) corresponding to the third-order differential operator $Z_2 \cdot N_1$ at the regular singular point $w = 1/4$. It is obvious that any linear combination of these series is also a solution of the differential equation. Consider the following combination instead of the third component in (16):

$$S_3^{(1/4)}(x) \longrightarrow (\ln(x/24) + 2/3) \cdot S_2^{(1/4)}(x) + S_{30}^{(1/4)}(x). \tag{20}$$

By writing the argument of the logarithm as $x/24$, there will be no logarithm in the connection matrix. Furthermore, by adding the second component of the basis to the third component with

a factor of $2/3$, the entry $(2, 2)$ of the connection matrix will be cancelled. The connection matrix then reads

$$C(0, 1/4) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -\dfrac{9}{64}\dfrac{\sqrt{3}}{\pi} \\ 0 & -\dfrac{3\pi\sqrt{3}}{32} & 0 \end{bmatrix}. \tag{21}$$

These tricks, based on well-chosen linear combinations of the solutions, allow us to obtain as many zeros as possible, and to get rid of the logarithms. They will be used in order to compute the connection matrix for L_6 between the point $w = 0$ and, respectively, $w = 1/4$, $w = -1/4$ and $w = \infty$.

The chosen well-suited basis of solutions, at each regular singular point calls for some comment. The factorization of the differential operator L_6 being $Y_3 \cdot Z_2 \cdot N_1$, our method of producing the solutions, sequentially, allows one to determine from which differential operator a given solution emerges. Near the points $w = 0$, $w = \pm 1/4$ and $w = \infty$, the third-order differential operator Y_3 brings three solutions (see table 1), one Frobenius series, one solution with a log term and one solution with a $\log^2$ term, denoted, respectively, by $\tilde{S}_4$, $\tilde{S}_5$ and $\tilde{S}_6$. The solutions of the differential operator Y_3 itself are of elliptic integral type (see appendix B). These elliptic integrals behave around $w = \pm 1/4$ (respectively $w = \infty$) like $g(t) \cdot \ln(t/16) + f(t)$, with $t = 1 - 16w^2$ (respectively $t = 1/16w^2$), $g(t)$ and $f(t)$ being series with rational coefficients. One may then assume that the logarithmic term that appears in the solutions of L_6, inherited from Y_3, will be of the form $\ln((1 - 16w^2)/16)$, near $w = \pm 1/4$, and of the form $\ln(1/256/w^2)$, near $w = \infty$. The general form of combination for the fourth to sixth components of the well-suited basis will be

$$\begin{aligned} &\tilde{S}_4 \longrightarrow \tilde{S}_4 \\ &\tilde{S}_5 \longrightarrow \tilde{S}_5 + (a_1 - \ln(c)) \cdot \tilde{S}_4 \\ &\tilde{S}_6 \longrightarrow \tilde{S}_6 + 2(a_1 - \ln(c)) \cdot \tilde{S}_5 + (\ln(c)^2 - 2a_1 \ln(c) + a_2) \cdot \tilde{S}_4 \end{aligned} \tag{22}$$

where $c = 1, 8, 16$ for the basis at, respectively, $w = 0$, $w = \pm 1/4$ and $w = \infty$. The values of the parameters a_1 and a_2 depend on each basis.

Note that the argument in $\ln(x/24)$ in the series solutions of the differential operator $Z_2 \cdot N_1$ at $w = 1/4$ will be $\ln(x/4)$ and $\ln(x/24)$ at, respectively, $w = \infty$ and $w = 1$. Similarly to Y_3, these arguments may come from the explicit solutions of Z_2.

3.2. The connection matrix between $w = 0$ and $w = 1/4$

The first three local solutions at $w = 0$ are given by (11), (12), (13), and the fourth, fifth and sixth solutions read

$$\begin{aligned} S_4^{(0)}(x) &= [0, 1, 9, 34, 178, 692, \ldots], \\ S_5^{(0)}(x) &= S_4^{(0)}(x) \cdot \ln(x) + S_{50}^{(0)}(x) - S_4^{(0)}(x)/4, \\ S_6^{(0)}(x) &= S_4^{(0)}(x) \cdot \ln^2(x) + 2\big(S_{50}^{(0)}(x) - S_4^{(0)}(x)/4\big) \cdot \ln(x) \\ &\quad + S_{60}^{(0)}(x) - S_{50}^{(0)}(x)/2 + 25S_4^{(0)}(x)/16 \end{aligned}$$

with

$$\begin{aligned} S_{50}^{(0)}(x) &= [0, 0, 0, -2, 34, 241/3, \ldots], \\ S_{60}^{(0)}(x) &= [0, 0, 0, 0, -19/3, -7693/72, -575\,593/1800, \ldots]. \end{aligned}$$

At the singular point $w = 1/4$, we make use of the combination (22) which amounts to taking $x/8$ as an argument of the logarithms in the fourth, fifth and sixth component. The parameters a_1 and a_2 in (22) are, respectively, $23/6$ and $41/9$. The first three local series at $x = 1 - 4w$ are given in (14), (15), (17), (20) and the fourth, fifth and sixth read

$$\begin{aligned} S_4^{(1/4)}(x) &= [1, -1/8, 3/16, 29/512, \ldots], \\ S_5^{(1/4)}(x) &= (\ln(x/8) + 23/6) \cdot S_4^{(1/4)}(x) + S_{50}^{(1/4)}(x), \\ S_6^{(1/4)}(x) &= \left(\ln^2(x/8) + \tfrac{23}{3}\ln(x/8) + 41/9\right) \cdot S_4^{(1/4)}(x) \\ &\quad + 2(\ln(x/8) + 23/6) \cdot S_{50}^{(1/4)}(x) + S_{60}^{(1/4)}(x) \end{aligned} \tag{23}$$

with

$$\begin{aligned} S_{50}^{(1/4)}(x) &= [0, 457/480, -2231/1680, -128\,969/184\,320, \ldots] \\ S_{60}^{(1/4)}(x) &= [0, -967/100, 4312\,219/470\,400, 595\,578\,701/116\,121\,600, \ldots]. \end{aligned}$$

Connecting both solutions amounts to solving a linear system of 36 unknowns (the entries of the connection matrix). We have been able to recognize these entries which are obtained in the floating point form with a large number of digits. The connection matrix $C(0, 1/4)$ for the order 6 differential operator L_6 reads

$$C(0,1/4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -\dfrac{9\sqrt{3}}{64\pi} & 0 & 0 & 0 \\ 0 & -\dfrac{3\pi\sqrt{3}}{32} & 0 & 0 & 0 & 0 \\ 5 & \dfrac{1}{3} - 2 \cdot I_3^+ & \dfrac{3\sqrt{3}}{64\pi} & 0 & 0 & \dfrac{1}{16\pi^2} \\ -\dfrac{5}{4} & -\dfrac{3\pi\sqrt{3}}{32} & \dfrac{45\sqrt{3}}{256\pi} & 0 & \dfrac{1}{32} & 0 \\ \dfrac{29}{16} - \dfrac{2\pi^2}{3} & \dfrac{15\pi\sqrt{3}}{64} & -\dfrac{225\sqrt{3}}{1024\pi} - \dfrac{3\pi\sqrt{3}}{64} & \dfrac{\pi^2}{64} & 0 & 0 \end{bmatrix}. \tag{24}$$

Some comments on how these entries have been 'recognized' will be given below. Let us remark that, once the entries of the connection matrix have been obtained, a further change of basis can be made to get it as 'simple' as possible.

3.3. Connection matrices between $w = 0$ and the other regular singular points

The chosen basis of solutions and the connection matrices between $w = 0$ (high or low temperature) and, respectively, the anti-ferromagnetic point $w = -1/4$ and the point $w = \infty$ (corresponding to $s = \pm i$) are given in appendix C.

The chosen basis, used for the regular singular points $w = 1, -1/2$ and $1+3w+4w^2 = 0$, are given in appendix D together with the corresponding connection matrices with the point $w = 0$. Many entries are 'recognized' and, in particular, those required to find the singular behaviour of the physical solution. They correspond to the third column of matrices given in appendix D.

The connection matrix between each pair of neighbouring singular points is computed with the well-defined procedure described above. The connection matrix between $w = 0$, and a non-neighbour singular point, is computed using (9). For instance, $C(0, 1)$ is computed from $C(0, 1/4)$ and $C(1/4, 1)$ as $C(0, 1) = C(0, 1/4) \cdot C(1/4, 1)$ which says that the solutions

defined at $w = 1/4$ connected to the solutions defined at $w = 0$, are also the solutions that are connected to the solutions defined at $w = 1$.

To be more confident of this prescription, let us underline that the connection matrices $C(0, 1)$ and $C(0, -1/2)$, deduced from (9), will be used below to confirm the known dominant singular behaviour of $\tilde{\chi}^{(3)}$ and find the subdominant behaviour.

3.4. Comments and remarks

The connection matrices between $w = 0$ and the other singular points are structured in blocks. The latter, due to the factorization of the differential operators and to the sequential building of the solutions, are easily recognized. The block $(1, 2, 3) \times (1, 2, 3)$ is associated with the third-order differential operator $Z_2 \cdot N_1$. The block $(4, 5, 6) \times (4, 5, 6)$ represents the connection between the solutions (at both $w = 0$ and the other singular points being considered) of L_6 that *are not* solutions of $Z_2 \cdot N_1$. The 'ferromagnetic constant' I_3^+ appears in the connection matrix between $w = 0$ and $w = 1/4$, as mentioned earlier, in the block $(1, 2, 3) \times (1, 2, 3)$ at the column corresponding to the $S_2^{(1/4)}$ (see (15)) solution of the third-order differential operator $Z_2 \cdot N_1$.

To compute the connection matrix, we have used the differential operator L_6 which has a unique factorization. If, instead, we consider the differential operator L_7, the next solution (around $w = 0$), which comes from M_1, will be the series (6) and will correspond to $\tilde{\chi}^{(3)}$. This seventh solution is expressed as a linear combination of the already existing components and of the solution of the differential operator L_1. We can then choose to add the latter as the seventh solution. The connection matrix will have an 1 at the entry $(7, 7)$ and zero elsewhere on the seventh line (and column), since the solution of the differential operator L_1 is global. By considering another factorization of L_7, we will get the same structure with an obvious relabelling.

Let us make a few computational remarks on the calculation of these connection matrices. At the matching of the series solutions for which 1500 coefficients[8] are generated from homogeneous and non-homogeneous recurrences, the entries of the matrix are computed with 800 digits for all the singular points. The numbers that come in the floating form are 'recognized' as powers of π, radicals and rational numbers, and are in agreement up to 400 digits[9] for the connection between the solutions at $w = 0$ and $w = \pm 1/4$, and up to 100 digits for the connection involving other singular points like $w = 1$. This fact is related to the convergence rate of the series at the (midway) chosen matching points. For instance, between $w = 0$ and $w = 1/4$, the matching points near $w = 1/8$ are such that both series (at $w = 0$ and $w = 1/4$), which have the same radius of convergence, will be faithfully reproduced with the number of terms used in the series. The matching of the solutions between $w = 1/4$ and $w = 1$, will then require more terms to fulfil the same accuracy than in the $(w = 0)$-$(w = 1/4)$ situation. This is due to the fact that, at $w = 1$, the convergence radius being $3/4$, the matching points, which should be in the common region of both discs, are closer to $w = 1/4$ than to $w = 1$. As a general rule, the matching points are chosen around the middle of the segment in the common area between the convergence discs of the two regular singular points for which the connection matrix is computed.

The difficulty in finding 'non-local' connection matrices is rooted in the recognition of the entries. We have given the connection matrix between $w = 0$ and $w = 1/4$ with entries fully recognized (apart from I_3^+) to show that the method actually works and is efficient. For the matrices concerning the connection between $w = 0$ and the other singular points, we have

[8] For some checks, 3000 terms have been generated.

[9] Let us note that the 'ferromagnetic constant' I_3^+ has been obtained up to more than 400 exact digits.

concentrated our effort on the entries that will show up in the physical solution. We should note that there is no reason to expect the other (not yet recognized) entries to be 'simply' combinations of π, log and radicals. These entries are probably *valuations of holonomic functions*. This was clearly seen in numerous examples we tackled of various differential equations (of order 2 and 3) with the known solutions of hypergeometric type. The recognition process used the fact that we actually found the explicit solutions of differential operator Y_3 and, thus, knew how the numerical logarithms can be tackled. These were 'absorbed' in the basis. We know, on the other hand, that the problem is strewn with hypergeometric functions. We then expect some π to be present. For the entries consisting of a simple product expression, recognizing the number amounts to performing simple arithmetic operations. Note that considering the inverse of the connection matrix, some entries also show up as simple rationals. The combination where π, radicals and rationals appear additively comes from looking, for instance, at the determinant of the matrices, or block matrices, which happen to be easily recognizable (in fact, rational or quadratic numbers for the roots of $1 + 3w + 4w^2 = 0$).

Another remark is the following. We first obtained the connection matrix (24) in some general basis. The matrix had more non-zero entries compared to (24) involving powers of π, radicals and also $\ln(3)$ and powers of $\ln(2)$. The well-suited basis we chose has 'evacuated' all these logs in the entries of the matrix, lessening the recognition-process effort. But, of course, all these logs will reappear in the final result such as the singular behaviour of the physical solution as next sections will show.

4. The physical solution $\tilde{\chi}^{(3)}$ and its singular behaviour

The calculations of connection matrices are obtained straightforwardly from the well-defined numerical process described in section 3. Having N singularities, one needs $N - 1$ such connection matrices in order to find the correspondence between all these well-suited bases of series solutions.

Let us focus on some particular entries of these various connection matrices, namely the entries corresponding to the decomposition of $\tilde{\chi}^{(3)}$ in terms of the various well-suited bases associated with each singularity. We have used the fact that the physical solution (corresponding to $\tilde{\chi}^{(3)}$) decomposes as the solution of differential operator $L_1, \mathcal{S}(L_1)$ (which is $\tilde{\chi}^{(1)}/2$) and the physical solution of the operator L_6 denoted by $\Phi_6(w)$ [8, 9]:

$$\tilde{\chi}^{(3)}(w) = \tfrac{1}{6}\tilde{\chi}^{(1)} + \Phi_6(w).$$

Furthermore, our well-suited basis of solutions at the singular point $w = 0$, does not contain, as a component, the physical solution $\Phi_6(w)$ which is given in terms of the previously considered components as

$$\Phi_6(w) = \tfrac{4}{3}S_1^{(0)} - \tfrac{1}{12}S_2^{(0)} - \tfrac{1}{4}S_4^{(0)}. \tag{25}$$

This physical solution can now be easily obtained from the connection matrices between $w = 0$ and any regular singular point, which we denote by $w = w_s$ (with $x = w$, $x = 1/w$ for, respectively, $w = 0$ and $w = \infty$ and $x = 1 - w/w_s$, otherwise) as

$$\Phi_6(x) = \sum_{j=1}^{6}\left(\frac{4}{3}C(0, w_s)_{1j} - \frac{1}{12}C(0, w_s)_{2j} - \frac{1}{4}C(0, w_s)_{4j}\right)\cdot S_j^{(w_s)}.$$

For instance, at the ferromagnetic critical point, this physical solution $\Phi_6(x)$ can easily be deduced from (24), and written as

$$\Phi_6(x) = -\frac{1}{4}\left(\frac{1}{3} - 2I_3^+\right)\cdot S_2^{(1/4)} - \frac{1}{64\pi^2}S_6^{(1/4)}.$$

Here $S_2^{(1/4)}$ and $S_6^{(1/4)}$ are known from their series expansion (15) and (23). This equation, giving the full expansion of $\tilde{\chi}^{(3)}$ at $w = 1/4$, can hardly be obtained directly from the integrals defining $\tilde{\chi}^{(3)}(w)$. One has similar expansions for all the other singular points.

4.1. Singular behaviour of $\tilde{\chi}^{(3)}$

Knowing the behaviour of solutions $S_j^{(w_s)}$ near each regular singular point, it is straightforward to get the singular behaviour at those points for the physical solution Φ_6 (and thus $\tilde{\chi}^{(3)}$).

Considering the critical behaviour of $\tilde{\chi}^{(3)}$ near the ferromagnetic critical point $w = 1/4$, and denoting $x = 1 - 4w$, the singular part of the 'physical' solution $\tilde{\chi}^{(3)}$ reads

$$\tilde{\chi}^{(3)}(\text{singular}, 1/4) = \frac{1}{2}\frac{I_3^+}{x} - \frac{1}{64\pi^2} S_4^{(1/4)} \cdot \ln^2(x) + \frac{1}{32\pi^2}\left(\left(3\ln(2) - \frac{23}{6}\right) \cdot S_4^{(1/4)} - S_{50}^{(1/4)}\right) \cdot \ln(x) \tag{26}$$

where I_3^+ is actually the 'ferromagnetic constant' (2), and $S_i^{(1/4)}$ the series defined in the well-suited basis (23) at $w = 1/4$. The results agree with previous results of Nickel, but the correction terms are new[10], in particular the term $3\ln(2)/32/\pi^2$ in (26). In terms of the $\tau = (1/s - s)/2$ variable introduced in [6, 15, 16], the singular part (26) reads

$$\tilde{\chi}^{(3)}(\text{singular}, \tau \simeq 0) \simeq \frac{I_3^+}{\tau^2} - \frac{\ln^2(\tau)}{16\pi^2} + \left(\ln(2) - \frac{23}{24}\right) \cdot \frac{\ln(\tau)}{4\pi^2} + \cdots.$$

Near the anti-ferromagnetic critical point $w = -1/4$, $\tilde{\chi}^{(3)}$ behaves as

$$\tilde{\chi}^{(3)}(\text{singular}, -1/4) = -\frac{1}{32\pi^2} S_4^{(-1/4)} \cdot \ln^2(x) - \frac{1}{16\pi^2}\left(3(2 - \ln(2)) \cdot S_4^{(-1/4)} + S_{50}^{(-1/4)}\right) \cdot \ln(x). \tag{27}$$

At the non-physical singularities $w = 1$ and $w = -1/2$ the physical solution behaves, respectively, like

$$\tilde{\chi}^{(3)}(\text{singular}, 1) = \frac{\sqrt{3}}{27\pi} \cdot S_2^{(1)} \cdot \ln(x) \tag{28}$$

and

$$\tilde{\chi}^{(3)}(\text{singular}, -1/2) = -\frac{8\sqrt{3}}{27\pi} \cdot S_2^{(-1/2)} \cdot \ln(x) \tag{29}$$

confirming Nickel's calculations given in [7].

At the point $w = \infty$, corresponding to the non-physical singularities $s = \pm i$, the singular behaviour reads

$$\tilde{\chi}^{(3)}(\text{singular}, \infty) = -\frac{1}{16\pi^2} S_4^{(\infty)} \cdot \ln^2(x) - \frac{1}{8\pi^2}\left((4 - 2\pi i) \cdot S_2^{(\infty)} - \left(5 + 4\ln(2) + i\frac{\pi}{2}\right) \cdot S_4^{(\infty)} + S_{50}^{(\infty)}\right) \cdot \ln(x). \tag{30}$$

At the new singularities found in [8], namely the roots of $1 + 3w + 4w^2 = 0$, which are regular singular points of the differential equation, the singular part of the physical solution reads, at first sight,

$$\tilde{\chi}^{(3)}(\text{singular}, w_1) = -\tfrac{1}{12}(a_{23} + 3a_{43}) \cdot S_2^{(w_1)} \cdot \ln(x).$$

The entries a_{23} and a_{43} (see the connection matrix for these points in appendix D) are, however, such that $a_{23} + 3a_{43} = 0$. The physical solution is thus, not singular, at the newly found quadratic singularities, confirming our conclusion given in [9] from series analysis.

[10] These results have also been found by B Nickel (private communication).

4.2. Asymptotic series analysis

As the physical solution $\tilde{\chi}^{(3)}$ is given as a series around $w = 0$, the coefficients of the latter are controlled by the nearest singular points (i.e., $w = \pm 1/4$). Since the singular parts at the ferromagnetic and anti-ferromagnetic critical points (26), (27) are obtained, it is straightforward to deduce the behaviour of the coefficients of series (6) for large values of n. Standard study of the asymptotic behaviour of the coefficients via their linear recursion relation can be used (see [17]). For our purpose, we use the following identity for $\ln^2(1-x)$ (where x stands for $x = 4w$),

$$\ln^2(1-x) = \sum_{n=2}^{\infty} b(n) \cdot x^n, \qquad \text{where} \quad b(n) = \sum_{i=1}^{n-1} \frac{1}{i(n-i)} = \frac{2}{n} \cdot (\Psi(n) + \gamma) \tag{31}$$

where $\gamma = 0.577\,215\,66\ldots$ denotes Euler's constant, and Ψ denotes the logarithmic derivative of the Γ function. Recalling the asymptotic expansion of $\Psi(n)$ up to $1/n^2$ for large values of n, one obtains

$$b(n) \to \frac{2}{n} \cdot \left(\gamma + \ln(n) - \frac{1}{2n} - \frac{1}{12n^2} + \cdots\right).$$

With the same manipulations of $\ln^2(1+x)$, and inserting in (26), (27), one obtains the asymptotic form of coefficients of $\tilde{\chi}^{(3)}/8w^9$ as

$$2^{-15} \cdot \frac{c(n)}{4^n} \simeq \frac{I_3^+}{2} - \frac{1}{16\pi^2}\left(\frac{1}{2} + (-1)^n\right)\left(\frac{\ln(n)}{n} + \frac{b_1}{n} - \frac{1}{2n^2}\right) + \frac{1}{16\pi^2}\left(\frac{23}{12} + 6(-1)^n\right)\frac{1}{n} + \cdots$$

where $b_1 = \gamma + 3\ln(2)$.

It is this parity effect in the asymptotic behaviour of the coefficients that we saw, numerically, (see equations (33) in [9]) where we obtained, around $n \simeq 500$, $c(n) \simeq 13.5 \times 4^n$ for n even and $c(n) \simeq 11 \times 4^n$ for n odd. For very large values of n, the asymptotic value of the coefficient $c(n)/4^n$ is thus $2^{14} \cdot I_3^+ \simeq 13.344\,154\,67\ldots$.

5. Monodromy matrices for $\tilde{\chi}^{(3)}$

5.1. Sketching the differential Galois group of L_7

As a consequence of the direct sum (8), the differential Galois group of L_7 reduces (up to a product by $\mathcal{C}$) to the differential Galois group of L_6. From the factorization of L_6, one can immediately deduce that the differential Galois group of L_6 is the semi-direct product of the differential Galois group of Y_3, of the differential Galois group of Z_2 and of the differential Galois group of N_1 (namely $\mathcal{C}$).

In some 'well-suited global basis' of solutions, the form of the 6×6 matrices representing the differential Galois group of L_6, reads

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{H} & \mathbf{B} \end{bmatrix}, \qquad \text{with} \qquad \mathbf{A} = \begin{bmatrix} b & 0 \\ \mathbf{h} & \mathbf{g} \end{bmatrix}$$

where the 2×2 matrix $\mathbf{g}$, and 3×3 matrix $\mathbf{B}$ correspond, respectively, to the differential Galois group of Z_2 and Y_3. The 3×3 matrix $\mathbf{A}$ is associated with the differential Galois group of $Z_2 \cdot N_1$, and the 3×3 matrix $\mathbf{H}$ corresponds to the fact that we have a *semi-direct* product of the differential Galois group of Y_3 and $Z_2 \cdot N_1$ in $L_6 = L_3 \cdot Z_2 \cdot N_1$.

Many papers (for instance [18–22]) describe how to calculate the differential Galois groups of order 2 and order 3 differential operators. The differential Galois group of L_7 will be deduced in a forthcoming publication [14].

To go beyond this sketchy description of the differential Galois group, one needs to calculate specific elements like the monodromy matrices expressed in a common basis.

5.2. Monodromy matrices rewritten in the $w = 0$ basis

Having the connection matrices between $w = 0$ and each singularity, the local monodromy matrices expressed in their own well-suited basis of (series) solutions, can be rewritten in a *unique* global basis valid for all singularities. This will allow us, in a second step, to calculate their products and thus generate the differential Galois group. Let us define the 2×2 and 3×3 matrices

$$A = \begin{bmatrix} 1 & 0 \\ \Omega & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ \Omega & 1 & 0 \\ \Omega^2 & 2\Omega & 1 \end{bmatrix} \tag{32}$$

where Ω denotes $2i\pi$ and corresponds to the translation of the logarithm when performing a complete rotation around the regular singular point: $\ln(w) \to \ln(w) + \Omega$.

The expression of the local monodromy matrix around each regular singular point w_s in its own well-suited basis of (series) solutions reads

$$l(w_s) = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & D \end{bmatrix} \tag{33}$$

where, ϵ and the 2×2 blocks C, and 3×3 blocks D, are such that

$$\begin{aligned} w = 0, w = \infty \quad &\to \quad \epsilon = +1, C = A, D = B \\ w = 1/4, \quad &\to \quad \epsilon = -1, C = A, D = B \\ w = -1/4, \quad &\to \quad \epsilon = -1, C = \mathrm{Id}, D = B \\ w = 1, -1/2, -3/8 \pm i\sqrt{7}/8, \quad &\to \quad \epsilon = +1, C = A, D = \mathrm{Id}. \end{aligned}$$

The monodromy matrix around any singularity $w = w_s$ expressed in terms of the $(w = 0)$ well-suited basis, and denoted as $M_{w=0}(w_s)$, reads

$$M_{w=0}(w_s) = C(0, w_s) \cdot l(w_s)(\Omega) \cdot C^{-1}(0, w_s). \tag{34}$$

In order to keep track of the π corresponding to the translation of the logarithm in the local monodromy matrix $l(w_s)(\Omega)$, and the π occurring in the expression of the entries of the (quite involved) connection matrix $C(0, w_s)$, we will denote the latter by $\alpha = 2i\pi$.

Let us focus on the singular point $w = 1$. Its monodromy matrix, expressed in terms of the $w = 0$ well-suited basis, is given by (34) with $w_s = 1$, and where the connection matrix $C(0, 1)$, matching the $(w = 1)$ well-suited basis together with the $(w = 0)$ well-suited basis, is a 'quite involved' matrix given in appendix D, with entries depending on π and on a set of 15 constants, not yet recognized in the closed form. The monodromy $M_{w=0}(1)$ can finally be written as a function of only α and Ω:

$$8\alpha^2 \cdot M_{w=0}(1)(\alpha, \Omega) = \begin{bmatrix} 8\alpha^2 & 0 & 0 & 0 & 0 & 0 \\ -48\,\alpha\,\Omega & 8\alpha^2 & -48\,\Omega & 0 & 0 & 0 \\ 0 & 0 & 8\alpha^2 & 0 & 0 & 0 \\ -1008\alpha\Omega & 0 & -1008\,\Omega & 8\alpha^2 & 0 & 0 \\ 12\,\alpha(5 + 16\alpha)\Omega & 0 & 12(5 + 16\alpha)\,\Omega & 0 & 8\alpha^2 & 0 \\ -\alpha(75 + 44\alpha^2)\,\Omega & 0 & -(75 + 44\alpha^2)\,\Omega & 0 & 0 & 8\alpha^2 \end{bmatrix}. \tag{35}$$

Let us give one more example corresponding to the new quadratic singularities $1+3w+4w^2=0$. The monodromy matrix around one of the quadratic singularities $w=w_1$, expressed in terms of the $(w=0)$ well-suited basis, after the conjugation (34), reads

$$8\alpha^2 \cdot M_{w=0}(w_1)(\alpha,\Omega) = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \tag{36}$$

with:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 8\alpha^2 & 0 & 0 \\ 48\alpha\,\Omega & 8\alpha(\alpha+6\,\Omega) & -144\,\Omega \\ 16\,\Omega\alpha^2 & 16\,\Omega\alpha^2 & -8\alpha(-\alpha+6\,\Omega) \\ -16\alpha\,\Omega & -16\alpha\,\Omega & 48\,\Omega \\ 4\alpha(4\alpha-15)\,\Omega & 4\alpha(4\alpha-15)\,\Omega & -12(4\alpha-15)\,\Omega \\ \alpha a & \alpha a & -3a \end{bmatrix}$$

with $a=(-40\alpha+12\alpha^2+75)\,\Omega$ and $[C]=8\alpha^2\cdot\mathbf{Id}(\mathbf{3}\times\mathbf{3})$.

One can actually verify that the monodromy matrix around the other quadratic singularity $w=w_2$ (w_2 is the complex conjugate of w_1), expressed in terms of the $(w=0)$ well-suited basis, actually identifies with (36) where α has been changed into $-\alpha$.

We have totally similar results for all the other (regular) singularities. The expressions of the other monodromy matrices $M_{w=0}(w_s)$, around the other (regular) singular points $w=w_s$, are displayed in appendix E.

We have seen that the connection matrices depend on I_3^+ and on 'not yet recognized' (probably transcendental) numbers, like x_{42} and y_{41} (for the connection matrix between $w=0$ and $w=\infty$). Rewriting a monodromy matrix in a unique (global) basis like the $w=0$ basis, amounts to performing conjugation, like (34), of simple (local) monodromy matrices depending only on Ω, by these quite involved connection matrices. As a consequence, one does expect, at first sight, these monodromy matrices, rewritten in the unique $w=0$ basis, to be dependent on the still unknown numbers. For instance, one certainly expects the monodromy matrix around $w=1/4$ (see appendix E) to be expressed in terms of the transcendental number I_3^+, or the monodromy matrix (35) to depend on 15 parameters. It is worth noting that all these matrices $M(w_s)$, expressed in the same $(w=0)$ well-suited basis, turn out to be quite simple matrices where the entries are actually rational expressions, with integer coefficients of α and Ω. Section 5.3 gives some hints on why this is so.

The introduction of the two parameters α and Ω is a nice 'trick' to track the π coming from the connection matrices versus the π coming from the local monodromy matrices. However, one should keep in mind that α is not independent of Ω: the 'true' monodromy matrices are such that $\alpha=\Omega$ (Ω being equal to $2i\pi$). Let us denote these 'true' monodromy matrices by M_i, $i=1,\ldots,8$:

$$\begin{aligned} M_1 &= M_{w=0}(\infty)(\Omega,\Omega), & M_2 &= M_{w=0}(1)(\Omega,\Omega), \\ M_3 &= M_{w=0}(1/4)(\Omega,\Omega), & M_4 &= M_{w=0}(w_1)(\Omega,\Omega), \\ M_5 &= M_{w=0}(-1/2)(\Omega,\Omega), & M_6 &= M_{w=0}(-1/4)(\Omega,\Omega), \\ M_7 &= M_{w=0}(0)(\Omega,\Omega), & M_8 &= M_{w=0}(w_2)(\Omega,\Omega). \end{aligned} \tag{37}$$

The matrices M_2, M_4, M_5, M_8, and respectively the matrices M_1 and M_7, share the same Jordan block form. The Jordan block forms for M_3 and M_6 read, respectively,

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

These matrices M_i are the generators of a 6×6 matrix representation of the differential Galois group of the Fuchsian differential equation corresponding to L_6. Any element of the differential Galois group is of the form

$$M_{P(1)}^{n_1} \cdot M_{P(2)}^{n_2} \cdot M_{P(3)}^{n_3} \cdot M_{P(4)}^{n_4} \cdot M_{P(5)}^{n_5} \cdot M_{P(6)}^{n_6} \cdot M_{P(7)}^{n_7} \cdot M_{P(8)}^{n_8} \tag{38}$$

where P denotes an arbitrary permutation of eight elements and where the n_i are positive or negative integers. This looks, at first sight, like an infinite discrete group, but the closure of this infinite set of matrices can be quite large continuous groups like semi-direct products of $SL(2, \mathcal{C})$ with $SL(3, \mathcal{C})$, and so on.

Our 'global' (800 digits, 1500 terms) calculations yield quite involved exact connection matrices. With such large and involved computer calculations there is always a risk of a subtle mistake or misprint. At this stage, and in order to be 'even more confident' in our results, let us recall that the monodromy matrices must satisfy one matrix relation which will be an *extremely severe non-trivial check* on the validity of these eight matrices M_i, or more precisely their (α, Ω) extensions. Actually, it is known (see, for instance, proposition 2.1.5 in [23]) that the monodromy group[11] of linear differential equations (with r regular singular points) is generated by a set of matrices $\gamma_1, \gamma_2, \ldots, \gamma_r$ that satisfy $\gamma_1 \cdot \gamma_2 \cdots \gamma_r = \mathbf{Id}$, where **Id** denotes the identity matrix. The constraint that 'some' product of all these matrices should be equal to the identity matrix, looks quite simple, but is, in fact, 'undermined' by subtleties of complex analysis on how connection matrices between non-neighbouring singular points should be computed. The fact that the prescription (9), (10) has given no contradictory results on the $\tilde{\chi}^{(3)}$ singular behaviour may be an argument that our M_i are not 'too far' from these 'elementary' γ_i. In other words, one of the products (38) must be equal to the identity matrix for some set of n_i and for some permutation P. With the particular choice (37) of ordering of the eight singularities, this product actually reads

$$M_1 \cdot M_2 \cdot M_3 \cdot M_4 \cdot M_5 \cdot M_6 \cdot M_7 \cdot M_8 = \mathbf{Id}. \tag{39}$$

Of course, from this relation, one also has seven other relations deduced by cyclic permutations. It is important to note that these relations (39) are not verified by extensions like (35), (36) depending on two independent parameters α and Ω, of the monodromy matrices M_i. If one imposes relations (39) for the (α, Ω) extensions of the M_i, one will find that, necessarily, α has to be equal to Ω, but (of course[12]) one will find that these matrix identities are verified for any value of Ω, not necessarily equal to $2i\pi$.

5.3. Comments

The entries of the connection matrices are seen to be expressed as various polynomials, or algebraic combinations of powers of π, $\ln(2)$, $\ln(N)$ (N integer), algebraic numbers, etc, and more 'involved' transcendental numbers like (2). On the other hand, the monodromy matrices

[11] Which identifies in our Fuchsian case with the differential Galois group.

[12] A matrix identity like (39) yields a set of polynomial (with integer coefficients) relations on $\Omega = 2i\pi$. The number π being transcendental it is not a solution of a polynomial with integer coefficients. These polynomial relations have, thus, to be polynomial identities valid for any Ω.

$M_{w=0}(w_s)$, expressed in the same $(w = 0)$ well-suited bases, have entries which are rational expressions with integer coefficients of α and Ω. To get some hint as to how this occurs, let us consider, for instance, the regular singular point $w = 1$. The local monodromy matrix is almost the unity matrix (only one solution with log) with elements

$$l(1)_{ij} = \delta_{ij} + \Omega \cdot \delta_{i3}\delta_{j2}. \tag{40}$$

The product (34) giving the global monodromy matrix will be given by

$$M_{w=0}(1)_{ij} = \delta_{ij} + \Omega \cdot C(0,1)_{i3} \cdot C^{-1}(0,1)_{2j} \tag{41}$$

where one can see that only the third column of $C(0, 1)$ and the second row of its inverse will contribute. These entries have been 'recognized' (see appendix D).

Let us assume that there is another solution with a log term (this is not so, see table 1). An entry (for instance $l(1)_{65}$) of the local monodromy matrix changes from zero to Ω. In this case equation (41) becomes

$$M_{w=0}(1)_{ij} = \delta_{ij} + \Omega \cdot C(0,1)_{i3}C^{-1}(0,1)_{2j} + \Omega \cdot C(0,1)_{i6}C^{-1}(0,1)_{5j}.$$

The entries $C(0, 1)_{i6}$ and $C^{-1}(0, 1)_{5j}$ will appear in the global monodromy matrix. In fact, changing the entry $l(1)_{65}$ from zero to Ω means that a formal solution will exhibit logs, and this will correspond to the entries $C(0, 1)_{i6}$. As a practical rule, we found that such entries (corresponding to solutions with logs) can be easily 'recognized' in contrast with the entries corresponding to Frobenius series which will be cancelled by the zero entries of $l(1)$. The entries corresponding to Frobenius series are probably valuations of holonomic functions.

Let us now assume (for the actual situation) that the whole column $C(0, 1)_{i3}$ has unknown entries. Recalling the fact that the product of the monodromy matrices, expressed in the same basis, should be equal to the identity matrix [23] (this is what we found for our eight matrices M_i, see (39)), one then expects the 'not yet guessed constants' (i.e., the column $C(0, 1)_{i3}$) to be given by a non-linear system of equations. This is indeed what occurs for this example, and we recover that way the entries given for this case in appendix D.

A last remark is the following. Right now, we have considered all the matrices (connection and therefore monodromy matrices expressed in a unique basis) with respect to the $(w = 0)$ well-suited basis of solutions. This is motivated by the physical solution $\tilde{\chi}^{(3)}$ which is known as series around $w = 0$. In fact, we can switch to another $w = \tilde{w}$ well-suited basis of solutions. This amounts to considering the connection $C(\tilde{w}, w_s) = C^{-1}(0, \tilde{w}) \cdot C(0, w_s)$. For instance, we have actually performed the same calculations for the $(w = 1/4)$ basis of series solutions. We have calculated all the connection matrices from the $(w = 1/4)$ basis to the other singular point basis series solutions, and deduced the exact expressions of the corresponding monodromy matrices now expressed in the same $(w = 1/4)$ basis of series solutions. It is worth noting that we get, this time, for the monodromy $M_{w=1/4}(w_s)$ around singular point w_s and expressed in the $(w = 1/4)$ basis, a matrix whose entries depend rationally on α, Ω, but, this time, also (except for the monodromy matrix at $w = 1$) on the 'ferromagnetic constant' I_3^+. One verifies that the product of these monodromy matrices in the *same order* as (39), is actually equal to the identity matrix when $\alpha = \Omega$, the matrix identity being valid for any value of $\alpha = \Omega$ (equal or not to $2i\pi$), and for any value of I_3^+ (equal or not to its actual value given in (2)).

We have similar results for the monodromy matrices around singular point w_s, expressed in the $(w = \infty)$ basis, but, now, the monodromy matrices $M_{w=\infty}(w_s)$ depend on α, Ω, and, this time, on the (not yet recognized) constants y_{41} and x_{42}. Again, the product of these monodromy matrices in the same order as (39), is actually equal to the identity matrix when $\alpha = \Omega$, the matrix identity being valid for any value of $\alpha = \Omega$ (equal or not to $2i\pi$) and for any values of y_{41} and x_{42} (equal, or not, to their actual values given in appendix C).

6. *Mutatis mutandis*: connection matrices and singular behaviour for $\tilde{\chi}^{(4)}$

6.1. Connection matrices

The Fuchsian differential equation for[13] $\tilde{\chi}^{(4)}$, the 4-particle contribution to the susceptibility, is given in [10]. The order 10 differential operator $\mathcal{L}_{10}$ associated with this differential equation has 36 (equivalent up to isomorphisms) factorizations (see appendix F in [10]). Consider, for instance, two of these factorizations:

$$\begin{aligned}\mathcal{L}_{10} &= N_8 \cdot M_2 \cdot L_{25} \cdot L_{12} \cdot L_3 \cdot L_0 \\ &= M_1 \cdot L_{24} \cdot L_{13} \cdot L_{17} \cdot L_{11} \cdot N_0. \end{aligned} \tag{42}$$

The notations are the same as those in [10], the M operators are of order 4, the N and L operators are, respectively, of order 2 and 1. The two factorizations above mean that $\mathcal{L}_{10}$ is a *direct sum* of an order 8 differential operator, $\mathcal{L}_8 = M_2 \cdot L_{25} \cdot L_{12} \cdot L_3 \cdot L_0$ and of the order 2 differential operator N_0 (which, see [10], has remarkably $\tilde{\chi}^{(2)}$ as solution):

$$\mathcal{L}_{10} = \mathcal{L}_8 \oplus N_0. \tag{43}$$

As was the case for $\tilde{\chi}^{(3)}$, it is thus sufficient to consider the differential operator $\mathcal{L}_8$ for which a general form of 8×8 matrices, representing $\mathcal{G}al(\mathcal{L}_8)$, the differential Galois group of $\mathcal{L}_8$, is deduced:

$$\begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{G} & \mathbf{M} \end{bmatrix}.$$

Here $\mathbf{G}$, $\mathbf{M}$ and $\mathbf{L}$ are 4×4 matrices, the latter being lower triangular. Recall that $\mathcal{L}_8$ has four known global solutions (see [10] and below).

Similarly to the calculation on $\tilde{\chi}^{(3)}$, we can, for instance, calculate connection matrices associated with the correspondence between the series near $x = 16w^2 = 0$ (high temperature) with the series near $x = 16w^2 = 1$ (the ferromagnetic and anti-ferromagnetic critical point), and find how the 'physical solution' $\tilde{\chi}^{(4)}$ can be decomposed on the various well-suited bases around each singular point (physical or non-physical) of the order 10 Fuchsian differential equation.

We use the factorization (42) to construct the basis of solutions, sequentially, as the four solutions corresponding to the differential operator $L_{25} \cdot L_{12} \cdot L_3 \cdot L_0$ that we call respectively S_1, S_2, S_3 and S_4. To these solutions, we add the four solutions coming from $\mathcal{L}_8$ and inherited from the differential operator M_2, that we call $S_5, \ldots, S_8$. Here, again, an optimal choice of basis is made in order to have as many zeros as possible in the connection matrix with as 'simple' entries as possible. The bases of solutions at $x = 0$ and $x = 1$ (with, respectively, $t = x$ and $t = 1 - x$) have similar forms and read

$$\begin{aligned}
&S_1(t) = 1, \qquad S_2(t) = \text{equation (33) in [10]}, \\
&S_3(t) = \text{equation (32) in [10]}, \qquad S_4(t) = \text{equation (43) in [10]}, \\
&S_5(t) = \text{see below}, \qquad S_6(t) = S_5(\ln(t/16) + a_1) + S_{60} \\
&S_7(t) = (\ln(t/16)^2 + 2a_1 \ln(t/16) + a_2) \cdot S_5 + 2S_{60}(\ln(t/16) + a_1) + S_{70} \\
&S_8(t) = (\ln(t/16)^3 + 3a_1 \ln(t/16)^2 + 3a_2 \ln(t/16) + a_3) \cdot S_5 \\
&\qquad + 3(\ln(t/16)^2 + 2a_1 \ln(t/16) + a_2) \cdot S_{60} + 3(\ln(t/16) + a_1) \cdot S_{70} + S_{80}
\end{aligned}$$

[13] $\tilde{\chi}^{(n)}$ is defined as $\chi^{(n)} = (1 - s^{-4})^{1/4} \cdot \tilde{\chi}^{(n)}$, for n even.

where the constants a_1, a_2 and a_3 and the series read, near $x = 0$,

$$a_1 = 79/60, \qquad a_2 = -751/1800, \qquad a_3 = -10\,619/375,$$
$$S_5^{(0)}(t) = [0, 0, 1, 45/32, 425/256, 945/512, \ldots],$$
$$S_{60}^{(0)}(t) = [0, 2/3, 0, 2353/13\,440, 121\,619/322\,560, \ldots],$$
$$S_{70}^{(0)}(t) = [8, -119/45, 0, -560\,333/1411\,200, \ldots],$$
$$S_{80}^{(0)}(t) = [0, 0, 0, 0, -127\,639\,044\,817/85\,349\,376\,000, \ldots]$$

and, near $x = 1$,

$$a_1 = 35/6, \qquad a_2 = 107/9, \qquad a_3 = -1051\,745\,657/749\,700$$
$$S_5^{(1)}(t) = [1, -1/4, -7/64, -45/256, -3385/16\,384, \ldots],$$
$$S_{60}^{(1)}(t) = [0, 7/120, -3809/13\,440, 42\,401/16\,120, 9271\,027/18\,923\,520, \ldots],$$
$$S_{70}^{(1)}(t) = [0, 1099/75, 741\,847/78\,400, 218\,499\,331/101\,606\,400, \ldots],$$
$$S_{80}^{(1)}(t) = [0, 0, 0, -37\,462\,660\,457/592\,220\,160, \ldots].$$

The connection matrix between $x = 0$ and $x = 1$ comes out as

$$C(0, 1) = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{A} & \mathbf{B} \end{bmatrix} \tag{44}$$

where $\mathbf{1}$ denotes the 4×4 identity matrix and $\mathbf{0}$ denotes the 4×4 zero matrix. The 4×4 identity matrix corresponds to the fact the four solutions $S_1, \ldots, S_4$ are *global solutions*. The two lower 4×4 blocks read

$$\mathbf{A} = \begin{bmatrix} a_{51} & a_{52} & -\frac{5}{2} & a_{54} \\ 0 & \frac{2}{3}\pi & 0 & \frac{1}{32} \\ a_{71} & 0 & a_{73} & 0 \\ a_{81} & -\pi^3 & a_{83} & a_{84} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2\pi^3} \\ 0 & 0 & -\frac{1}{2\pi} & 0 \\ 0 & -\frac{\pi}{2} & 0 & 0 \\ -\frac{\pi^3}{2} & 0 & 0 & 0 \end{bmatrix}$$

with

$$a_{71} = \frac{\pi^2}{6} - \frac{2422}{225}, \qquad a_{73} = \frac{5\pi^2}{6} + \frac{2422}{225}, \qquad a_{84} = -\frac{\pi^2}{32} - \frac{1211}{600}.$$

The 'not yet recognized' entries of this matrix read

$$a_{51} \simeq -17.882\,936\,774\,520, \qquad a_{52} \simeq 7.767\,669\,067\,696, \qquad a_{54} \simeq 0.530\,951\,641\,617,$$
$$a_{81} \simeq -92.773\,462\,923\,758, \qquad a_{83} \simeq 77.887\,072\,991\,056.$$

Here again, the block structure of the connection matrix relies on the factorization of $\mathcal{L}_8$ and on the 'sequential' building of the solutions. The block matrix $\mathbf{B}$ represents, specifically, the connection between the solutions inherited from M_2 at both points $x = 0$ and $x = 1$. This fourth-order differential operator M_2 in $\mathcal{L}_8$ (corresponding to $\tilde{\chi}^{(4)}$) is structurally very similar (see the remark at end of appendix B) to operator Y_3 in L_6 ($\tilde{\chi}^{(3)}$). Similarly to the $\tilde{\chi}^{(3)}$ case, a ferromagnetic (and anti-ferromagnetic) constant (see (48) below) is localized at the fifth line.

We have also computed the connection matrices[14] (not given here) between the solutions at $x = 0$ and, respectively, $x = 4$ (corresponding to Nickel's non-physical singularities) and $x = \infty$ (corresponding to the non-physical singularities $s = \pm i$). Denoting by

[14] The matching points are taken in the lower half-plane of the variable x.

$M_{x=0}(0)$, $M_{x=0}(1)$, $M_{x=0}(4)$ and $M_{x=0}(\infty)$, the monodromy matrices expressed in the same $x = 0$ well-suited basis obtained with similar conjugation like (34), one obtains

$$M_{x=0}(\infty) \cdot M_{x=0}(4) \cdot M_{x=0}(1) \cdot M_{x=0}(0) = \mathbf{Id}. \tag{45}$$

This identity is valid irrespective of the still unknown constants.

6.2. *Singular behaviour of* $\tilde{\chi}^{(4)}$

The particular physical solution corresponding to $\tilde{\chi}^{(4)} = \tilde{\chi}^{(2)}/3 + \Phi_8$ (see [10]) is given, in terms of the basis chosen at the point $x = 0$, by

$$\Phi_8 = \frac{1}{384} \cdot \left(5S_1^{(0)} - 5S_3^{(0)} - 2S_5^{(0)}\right). \tag{46}$$

At the ferromagnetic, and anti-ferromagnetic, critical point $x = 1$, the solution can be deduced from the above connection matrix and reads

$$\Phi_8 = -\frac{1}{384}(2a_{51} - 5) \cdot S_1^{(1)} - \frac{a_{52}}{192} \cdot S_2^{(1)} - \frac{a_{54}}{192} \cdot S_4^{(1)} + \frac{1}{384\pi^3} \cdot S_8^{(1)}.$$

Here again, the above decomposition corresponds to an expansion at the point $x = 1$ of the triple integral defining $\tilde{\chi}^{(4)}$.

From this solution, the singular part of $\tilde{\chi}^{(4)}$ reads (with $t = 1 - x$)

$$\begin{aligned}\tilde{\chi}^{(4)}(\text{singular}, 1) = {} & \frac{I_4^-}{t} + \frac{1}{384\pi^3} S_5^{(1)} \cdot \ln^3(t) - \frac{1}{32\pi^3}\left(\left(\ln(2) - \frac{35}{24}\right) S_5^{(1)} - \frac{35}{24} S_{60}^{(1)}\right) \cdot \ln^2(t) \\ & + \frac{1}{8\pi^3} \cdot \left(\left(\ln(2)^2 - \frac{35}{12}\ln(2) + \frac{107}{144}\right) \cdot S_5^{(1)}\right. \\ & \left. - \left(\frac{1}{2}\ln(2) - \frac{35}{48}\right) \cdot S_{60}^{(1)} + \frac{1}{16} S_{70}^{(1)}\right) \cdot \ln(t) + \frac{1}{48\pi}\, {}_2F_1(1/2, -1/2; 2; t) \cdot \ln(t).\end{aligned} \tag{47}$$

The constant [1] I_4^- reads, in terms of the 'not yet recognized' numbers a_{52}, a_{54},

$$I_4^- = \frac{1}{36\pi} + \frac{a_{52}}{128} - \frac{a_{54}\pi}{24} \simeq 0.000\,025\,448\,511\,0658\ldots. \tag{48}$$

The first term on the right-hand side of (48) comes from $\tilde{\chi}^{(2)}$, as well as the last term in (47).

Similarly, the singular behaviour of the physical solution $\tilde{\chi}^{(4)}$ at the other singular points can easily be obtained from the corresponding connection matrices (not given here). At the singular point $x = 4$, the physical solution behaves like (with $t = 4 - x$)

$$\tilde{\chi}^{(4)}(\text{singular}, 4) = -\frac{\mathrm{i} \cdot t^{13/2}}{2^{10} \cdot 3^2 \cdot 5005}\left(1 + \frac{5}{4}t + \frac{261}{272}t^2 + \cdots\right) \tag{49}$$

confirming the calculations in [7].

The singular behaviour of $\tilde{\chi}^{(4)}$ at the singular point $x = \infty$ reads (with $t = 1/x$)

$$\begin{aligned}\tilde{\chi}^{(4)}(\text{singular}, \infty) = {} & -20\mathrm{i} \cdot t^{-1/2} \cdot \left(A_0 + 3A_1 \cdot \ln(t)\right. \\ & \left. + 3\left((a_1 - 4\ln(2)) \cdot S_5^{\infty} + S_{60}^{\infty}\right) \cdot \ln^2(t) + S_5^{\infty} \cdot \ln^3(t)\right) \\ & + \frac{(-t)^{-1/2}}{36\pi}\left(1 + \frac{3t}{4}\, {}_2F_1(1/2, 5/2; 2; t) \cdot \ln(-t) - \frac{9\pi t}{16}\sum_{n=0}^{\infty} b_n t^n\right)\end{aligned} \tag{50}$$

with

$$A_1 = \frac{2}{5}(2K-1)\cdot S_{41}^{\infty} + (16\ln^2(2) - 8a_1\ln(2) + a_2)\cdot S_5^{\infty} + 2(a_1 - 4\ln(2))\cdot S_{60}^{\infty} + 3S_{70}^{\infty}$$

$$\begin{aligned} A_0 = 2\pi^3 &\left(i_{52} + \mathrm{i}\frac{24}{\pi^2}(2K-1)\right) S_2^{\infty}/5 - \left(-\frac{48}{\pi^2}(2K-1) + \mathrm{i}(5+2r_{53})\right)\pi^3 S_3^{\infty}/5 \\ &- (64\ln^3(2) - 48a_1\ln^2(2) + 12a_2\ln(2) - a_3)\cdot S_5^{\infty} \\ &+ \frac{6}{5}(2K-1)\cdot S_{40}^{\infty} + 3(16\ln^2(2) - 8a_1\ln(2) + a_2)\cdot S_{60}^{\infty} \\ &+ 3(a_1 - 4\ln(2))\cdot S_{70}^{\infty} + S_{80}^{\infty} \end{aligned}$$

$$b_n = \frac{\Gamma(n+1/2)\Gamma(n+5/2)}{\Gamma(n+2)\Gamma(n+1)}\left(\Psi(n+2) + \Psi(n+1) - \Psi\left(n+\frac{5}{2}\right) - \Psi\left(n+\frac{1}{2}\right)\right)$$

where $K = 0.915\,965\ldots$ is *Catalan's constant* and the other parameters, constants and series are $a_1 = 2/5 - \pi\mathrm{i}$, $a_2 = 1 - \pi^2 - 4\pi\mathrm{i}/5$, $a_3 = -6\pi^2/5 + 481\,93/7500 + \pi(\pi^2 - 3)\mathrm{i}$, $i_{52} = -0.740\,250\,494\ldots$, $r_{53} = 2.225\,246\,651\ldots$, and

$$\begin{aligned} S_2^{\infty} &= \frac{1-6t+2t^2}{2(t-1)}, \qquad S_3^{\infty} = \frac{3-12t+8t^2}{8(t-1)^{3/2}} \\ S_{40}^{\infty} &= [2, 41/2, 313/48, 3047/480, \ldots], \\ S_{41}^{\infty} &= [1, -25/2, -61/8, -129/16, \ldots], \\ S_5^{\infty} &= [0, 1, 7/10, 47/64, 981/1280, \ldots], \\ S_{60}^{\infty} &= [0, 0, 161/300, 2039/4800, \ldots], \\ S_{70}^{\infty} &= [0, 0, 1847/180\,00, 2627/360\,00, \ldots], \\ S_{80}^{\infty} &= [0, 0, 0, 144\,238\,79/720\,000\,0, \ldots]. \end{aligned}$$

The last bracket in (50) comes from $\tilde{\chi}^{(2)}$.

Having the singular part of $\tilde{\chi}^{(4)}$ at the ferromagnetic and anti-ferromagnetic critical points, it is straightforward to obtain the asymptotic behaviour of the series coefficients. This time, one needs the form of the coefficients in the expansion of $\ln^3(1-x)$ that we find to be[15]

$$\ln^3(1-x) = \sum_{n=3}^{\infty}\left(-\frac{3}{n}(\Psi(n)+\gamma)^2 + \frac{\pi^2}{2n} - \frac{3}{n}\Psi(1,n)\right)\cdot x^n \tag{51}$$

where $\Psi(1,n)$ is the first derivative of $\Psi(n)$. Expanding $\Psi(n)$ and $\Psi(1,n)$ up to $1/n^2$ for large values of n, one obtains the following asymptotic behaviour for the coefficients of the $\tilde{\chi}^{(4)}$ series,

$$c(n) \simeq I_4^- - \frac{\ln^2(n)}{128\pi^3 n} + \frac{\ln(n)}{128\pi^3 n^2} - \frac{b_1\ln(n)}{64\pi^3 n} - \frac{b_2}{2304\pi^3 n} + \frac{b_1 - 1}{128\pi^3 n^2} + \cdots$$

where

$$\begin{aligned} b_1 &= \gamma + 4\ln(2) - \frac{35}{6}, \\ b_2 &= 288\ln^2(2) + 144\gamma\ln(2) + 18\gamma^2 - 210\gamma - 840\ln(2) + 45\pi^2 + 214. \end{aligned}$$

[15] An asymptotic form can be obtained using various packages available at http://algol.inria.fr/libraries/software.html like the command 'equivalent' in gfun [24], see details in [25, 26].

7. $\tilde{\chi}^{(1)} + \tilde{\chi}^{(3)}$ versus $\tilde{\chi}$ at scaling

Thus far we have discussed, in sections 4 and 6.2, the mathematical aspects of the solutions to the Fuchsian differential equations for $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$. However, the physics implications of the solutions we have obtained call for some remarks near the physical critical points. Taking, as an example, the ferromagnetic singularity for $\tilde{\chi}^{(3)}$, the sum of the first two n-particle terms behave at $\tau \simeq 0$ as

$$\tilde{\chi}^{(1)} + \tilde{\chi}^{(3)} \simeq \frac{1 + I_3^+}{\tau^2} - \frac{\ln^2(\tau)}{16\pi^2} + \left(\ln(2) - \frac{23}{24}\right) \cdot \frac{\ln(\tau)}{4\pi^2} + \frac{11}{48} + \frac{3}{8} I_3^+ - \frac{1}{4\pi^2}\left(\ln^2(2) - \frac{23}{12}\ln(2) + \frac{14}{144}\right) + \cdots . \tag{52}$$

The exact susceptibility, as reported in [16], yields for the normalized susceptibility $\tilde{\chi}$,

$$\tilde{\chi} = \frac{s}{(1-s^4)^{1/4}} \cdot \chi = \frac{(\tau + \sqrt{1+\tau^2})^{-1/2}}{(1+\tau^2)^{1/8}} \left(c_1 \tau^{-2} F_+(\tau) + \frac{\tau^{-1/4}}{\sqrt{2}} \sum_{p=0}^{\infty} \sum_{q=p^2}^{\infty} b_+^{(p,q)} \cdot \tau^q \ln^p(\tau) \right) \tag{53}$$

where $c_1 = 1.000\,815\,260\ldots$ is given with some 50 digits in [15]. $F_+(\tau)$ and $b_+^{(p,q)}$ are given in [15]. The constants $1 + I_3^+$ and c_1 verify $1 + I_3^+ + I_5^+ = c_1$ with nine digits, I_5^+, corresponding to $\chi^{(5)}$, is the constant given in [1] (and with some 30 digits in [6]). Thus, and as suggested in [1], the partial sums of the $\chi^{(n)}$ would converge rapidly to the full χ. Furthermore, adding $\chi^{(3)}$ term has resulted in a series expansion that reproduces the first 24 terms of χ to be compared with only eight first terms for $\chi^{(1)}$ series.

However, equation (53) shows a $\tau^{-1/4}$ divergence as an overall factor to the logarithmic singularities. This structure, absent in (52), could suggest, in the most pessimistic scenario, that the n-particle sequence is perhaps useless in understanding scaling corrections and that one should be cautious in accepting the conclusions of studies of higher field derivatives of the susceptibility, based on similar n-particle representations [27, 28]. The same situation occurs for the low-temperature regime when we compare the first two n-particle terms ($\tilde{\chi}^{(2)}$ and $\tilde{\chi}^{(4)}$) with the full $\tilde{\chi}$ at scaling[16].

This observation raises several profound issues, which we do not address here. One is how the logarithmic terms in the entire sum add up to make the $\tau^{-1/4}$ divergence be factored out. If one assumes that the other $\tilde{\chi}^{(2n+1)}$ terms share the same singularity structure as $\tilde{\chi}^{(3)}$, in particular the occurrence (in variable τ or s) of only *integer* critical exponents at the ferromagnetic critical point, the $\tau^{-1/4}$ divergence, as an overall factor, implies the following correspondence,

$$\sum_{n=1}^{\infty} \sum_{m=0}^{N(n)} \alpha_{n,m} \cdot S_{n,m}(\tau) \ln^m(\tau) \to \tau^{-1/4} \cdot \sum_{p=0}^{\infty} \sum_{q=p^2}^{\infty} b_+^{(p,q)} \tau^q \ln^p(\tau)$$

with $S_{n,m}(\tau)$ analytical at $\tau = 0$ and $\alpha_{n,m}$ numerical coefficients. $N(n)$ is the maximum power of logarithmic terms occurring in the solution around the ferromagnetic point of the differential equation of $\tilde{\chi}^{2n+1}$. This correspondence requires probably a *very particular structure* in the successive differential equations. Obtaining the differential equation for $\tilde{\chi}^{(5)}$ (or for $\tilde{\chi}^{(6)}$), and obtaining much larger series for the full susceptibility χ, will certainly help to guess such a structure and understand the susceptibility of the two-dimensional Ising model which

[16] For the leading amplitude, $\tilde{\chi}^{(2)}$ and $\tilde{\chi}^{(4)}$ give $1/12\pi + I_4^- \simeq 1.000\,9593\ldots/12\pi$ which is very close to $1.000\,9603\ldots/12\pi$ for the full $\tilde{\chi}$ [6].

continues to be a treasure-trove of profound insights into both the mathematics and physics of integrable systems.

Let us note that the phenomenon we have discussed may be more widespread than that observed here. If so, a whole new chapter could be opened on field-theoretical expansions. The challenging problem one faces here is to link *linear and non-linear* descriptions of a physical problem, namely the description in terms of an infinite number of holonomic (linear) expressions for a physical quantity of a non-linear nature. Actually, the latter is 'Painlevé like' since its series expansion can be obtained from a program of *polynomial growth* which uses exclusively a quadratic finite difference double recursion generalizing the Painlevé equations [15, 16]. The difficulty to link holonomic versus non-linear descriptions of physical problems is typically the kind of problems one faces with the Feynmann diagram approach of particle physics, but the susceptibility of the Ising model is, obviously, the simplest non-trivial example to address such an important issue.

8. Conclusion

We have introduced a simple and very efficient method to calculate numerically, with an arbitrary number of digits, the connection matrices between the independent solutions, defined at two singular points, of differential equations of quite high orders. We have considered the order 7, and 10, Fuchsian ODEs corresponding to the 3- and 4-particle contribution to the magnetic susceptibility of the Ising model. The entries of the connection matrix between two regular singular points have been obtained in the floating point form and most of them have been recognized, particularly those that show up in the singular behaviour of the physical solutions. They are expressed as polynomial, or algebraic, combinations of π, $\ln(2), \ldots,$ radicals, and more involved numbers (not yet recognized) such as the 'ferromagnetic constant' (2). The method allows us to obtain the series expansions of the physical solutions $\tilde{\chi}^{(3)}$ (and $\tilde{\chi}^{(4)}$) around any other regular singular point, besides the already known series around $w = 0$. We obtained, in this way, near each singular point all the dominant and subdominant singular behaviours of the physical solutions. Such subdominant singular behaviour is certainly hard to obtain from series analysis. At the newly found quadratic singularities of the differential equation, we showed that the physical solution $\tilde{\chi}^{(3)}$ itself *is not singular*. Also note, at $w = 1/4$, that the behaviour in $(1 - 4w)^{-3/2}$ corresponding to the largest critical exponent for the ODE is *actually absent* in the physical solution. Note the remarkable fact that the factorization of differential operator L_7 (and $\mathcal{L}_{10}$) associated with $\tilde{\chi}^{(3)}$ (respectively $\tilde{\chi}^{(4)}$) shows clearly the differential operator responsible for the non-physical singularities given in [6, 7] and the newly found quadratic numbers [8]. In both cases ($\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$), these non-physical singularities are carried by the differential operator $Z_2 \cdot N_1$ (respectively $L_{25} \cdot L_{12} \cdot L_3 \cdot L_0$) occurring at the right of L_7 (respectively $\mathcal{L}_{10}$).

The physical solutions $\tilde{\chi}^{(3)}$ (and $\tilde{\chi}^{(4)}$) being known as series around $w = 0$, the growth behaviour of the corresponding series coefficients should be controlled by the singular behaviour at the nearest singular points which are the ferromagnetic and anti-ferromagnetic critical points in both cases ($w = \pm 1/4$ and $x = 1$). This growth is easily found from the expansion around the ferromagnetic and anti-ferromagnetic points.

The connection matrices we have obtained allow us to relate the solutions around any given singular point to a common (non-local) basis of solutions. In this respect, we have obtained the exact expression of all the monodromy matrices, expressed in the *same* basis, and we have seen that they are simple matrices with rational function entries. In a forthcoming publication [14], we will give the whole structure of the differential Galois group for the two previous Fuchsian differential equations.

As far as the physics implications of the solutions are concerned, we have compared the corrections to scaling at the ferromagnetic point given by the first two terms ($\chi^{(1)}$ and $\chi^{(3)}$) with the full χ. Qualitative difference is found raising profound issues on the n-particle representation of the susceptibility. The same observation occurs for the anti-ferromagnetic point, and also for the low-temperature regime.

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Note added in proof. After completion of the revised version of our manuscript we were told that, as a consequence of the work of McCoy, Tracy and Wu, the two transcendental numbers I_3^+ and I_4^- can actually be written in terms of polylogarithms, namely the Clausen function Cl_2 and of the Riemann zeta function, as follows:

$$I_3^+ = \frac{1}{2\pi^2}\cdot\left(\frac{\pi^2}{3} + 2 - 3\sqrt{3}\cdot Cl_2\left(\frac{\pi}{3}\right)\right), \qquad Cl_2(\theta) = \sum_{n=1}^{\infty}\frac{\sin(n\,\theta)}{n^2}$$

$$I_4^- = \frac{1}{16\pi^3}\cdot\left(\frac{4\pi^2}{9} - \frac{1}{6} - \frac{7}{2}\cdot\zeta(3)\right).$$

The derivation of these results has never been published but these results appeared in a conference proceedings [30]. We have actually checked that I_3^+ and I_4^- we got from the calculations displayed in our paper as floating numbers with respectively 450 digits and 400 digits accuracy *are actually in agreement* with the previous two formula. These two results provide a clear answer to the question of how 'complicated and transcendental' some of our constants occurring in the entries of the connection matrices can be. These extremely interesting results are not totally surprising when one recalls the deep link between zeta functions, polylogarithms and hypergeometric series [31–34].

Appendix A

We give, in this appendix, the explicit expressions of the differential operators X_1 and Z_2 and Y_3. The order 1 differential operator X_1 reads

$$X_1 = \frac{\mathrm{d}}{\mathrm{d}w} + \frac{p_0}{p_1} \tag{A.1}$$

with

$$\begin{aligned} p_1 = {} & (-1+w)(4w-1)(1+2w)(4w+1)(1+3w+4w^2)(1-3w-18w^2+104w^3+96w^4) \\ & \times (1-7w-4w^2-47w^3+36w^4+280w^5+160w^6+256w^7) \\ p_0 = {} & w\cdot(-58+909w+3284w^2-247\,11w^3-723\,52w^4+181\,016w^5 \\ & +125\,176\,8w^6+285\,288\,0w^7+145\,459\,2w^8-114\,556\,16w^9 \\ & -317\,122\,56w^{10}-204\,185\,60w^{11}+208\,404\,48w^{12}+349\,634\,56w^{13} \\ & +301\,465\,60w^{14}+157\,286\,40w^{15}). \end{aligned}$$

The order 2 differential operator Z_2 is

$$Z_2 = \frac{1}{p_2}\sum_{n=0}^{2} p_n\cdot\frac{\mathrm{d}}{\mathrm{d}w^n} \tag{A.2}$$

where the polynomials p_i, now, read

$$\begin{aligned}
p_2 &= w\cdot(4w-1)^2(4w+1)(1+3w+4w^2)(-1+w)(1+2w)\\
&\quad\times(1-3w-18w^2+104w^3+96w^4)\\
p_1 &= (4w-1)(1-6w-111w^2-108w^3+1080w^4-4488w^5\\
&\quad-403\,68w^6-942\,72w^7-483\,84w^8+727\,04w^9+491\,52w^{10})\\
p_0 &= 4+48w-276w^2-1520w^3-3192w^4-4224w^5-715\,52w^6\\
&\quad-307\,200w^7-239\,616w^8+983\,04w^9+983\,04w^{10}.
\end{aligned}$$

The order 3 differential operator Y_3 is given by

$$Y_3 = \frac{1}{p_3}\sum_{n=0}^{3} p_n\cdot\frac{\mathrm{d}^n}{\mathrm{d}w^n} \tag{A.3}$$

where the polynomials p_i now read

$$\begin{aligned}
p_3 &= w^2\cdot(w-1)(1+2w)(1+3w+4w^2)(4w-1)^3(4w+1)^3\\
&\quad\times(96w^4+104w^3-18w^2-3w+1)^3(1+19w-368w^2-3296w^3\\
&\quad+178\,82w^4+272\,599w^5+160\,900w^6-697\,920\,8w^7+755\,080\,0w^8\\
&\quad+203\,094\,872w^9-278\,920\,192w^{10}-395\,981\,430\,4w^{11}-211\,544\,742\,4w^{12}\\
&\quad+208\,947\,294\,72w^{13}+397\,197\,281\,28w^{14}+205\,160\,980\,48w^{15}\\
&\quad+256\,763\,363\,328w^{16}-327\,065\,010\,176w^{17}-881\,022\,776\,115\,2w^{18}\\
&\quad+414\,933\,057\,536w^{19}+116\,411\,936\,538\,624w^{20}+296\,827\,723\,186\,176w^{21}\\
&\quad+317\,648\,030\,138\,368w^{22}+179\,148\,186\,189\,824w^{23}\\
&\quad+194\,933\,533\,179\,904w^{24}+112\,931\,870\,081\,024w^{25}\\
&\quad-552\,461\,643\,284\,48w^{26}+110\,638\,357\,544\,96w^{27}+151\,182\,848\,819\,2w^{28})\\
p_2 &= w\cdot(4w-1)^2(4w+1)^2(96w^4+104w^3-18w^2-3w+1)^2\\
&\quad\times(6+102w-2018w^2-23\,962w^3+242\,904w^4+2\,575\,633w^5\\
&\quad-12\,389\,010w^6-178\,413\,527w^7+80\,727\,412w^8+6\,252\,221\,348w^9\\
&\quad+2\,456\,938\,016w^{10}-178\,278\,888\,104w^{11}-103\,902\,989\,696w^{12}\\
&\quad+3\,814\,815\,965\,856w^{13}+1\,524\,977\,514\,176w^{14}-67\,400\,886\,678\,400w^{15}\\
&\quad-74\,115\,827\,788\,032w^{16}+797\,710\,351\,468\,032w^{17}+2\,324\,376\,661\,856\,256w^{18}\\
&\quad-1\,561\,280\,104\,050\,688w^{19}-16\,314\,064\,973\,299\,712w^{20}\\
&\quad-27\,005\,775\,986\,622\,464w^{21}-40\,259\,640\,226\,480\,128w^{22}\\
&\quad+35\,764\,751\,009\,841\,152w^{23}+1\,007\,304\,244\,270\,727\,168w^{24}\\
&\quad+1\,460\,771\,505\,523\,654\,656w^{25}-13\,359\,756\,413\,056\,843\,776w^{26}\\
&\quad-63\,988\,213\,537\,189\,134\,336w^{27}-116\,684\,614\,339\,309\,600\,768w^{28}\\
&\quad-75\,710\,498\,024\,932\,245\,504w^{29}+57\,121\,462\,326\,803\,824\,640w^{30}\\
&\quad+132\,479\,693\,600\,191\,414\,272w^{31}+111\,232\,702\,128\,767\,107\,072w^{32}\\
&\quad+106\,152\,703\,871\,500\,156\,928w^{33}+83\,508\,376\,521\,540\,632\,576w^{34}\\
&\quad+10\,084\,606\,300\,752\,183\,296w^{35}-9\,404\,395\,631\,251\,816\,448w^{36}\\
&\quad+2\,682\,738\,003\,029\,262\,336w^{37}+297\,237\,575\,406\,452\,736w^{38})
\end{aligned}$$

$$
\begin{aligned}
p_1 = {} & 2\cdot(4w-1)(4w+1)(96w^4+104w^3-18w^2-3w+1)\\
& \times(-3-25w+1013w^2+7893w^3-353\,904w^4-1\,562\,671w^5\\
& +43\,285\,825w^6+192\,457\,911w^7-2\,690\,351\,207w^8-15\,077\,420\,736w^9\\
& +94\,510\,776\,436w^{10}+707\,838\,800\,508w^{11}-2\,327\,528\,107\,216w^{12}\\
& -23\,421\,365\,465\,744w^{13}+45\,755\,890\,012\,000w^{14}+568\,028\,144\,875\,200w^{15}\\
& -824\,814\,656\,530\,816w^{16}-10\,390\,722\,028\,797\,440w^{17}\\
& +12\,438\,134\,957\,505\,536w^{18}+145\,637\,031\,330\,319\,360w^{19}\\
& -127\,616\,737\,495\,506\,944w^{20}-1\,708\,173\,874\,007\,113\,728w^{21}\\
& -52\,355\,400\,373\,420\,032w^{22}+15\,741\,676\,181\,476\,802\,560w^{23}\\
& +24\,085\,046\,332\,129\,804\,288w^{24}-57\,977\,682\,482\,294\,161\,408w^{25}\\
& -168\,033\,877\,030\,234\,750\,976w^{26}-56\,941\,336\,876\,602\,621\,952w^{27}\\
& -426\,707\,803\,148\,891\,717\,632w^{28}-200\,805\,832\,817\,071\,095\,808w^{29}\\
& +8\,716\,841\,486\,700\,848\,873\,472w^{30}-6\,642\,009\,916\,749\,838\,811\,136w^{31}\\
& -192\,590\,979\,400\,145\,399\,971\,840w^{32}-564\,260\,086\,660\,360\,537\,374\,720w^{33}\\
& -585\,770\,764\,250\,229\,243\,904\,000w^{34}+235\,172\,208\,485\,444\,226\,121\,728w^{35}\\
& +1\,203\,159\,617\,695\,281\,059\,987\,456w^{36}+1\,323\,272\,087\,085\,206\,269\,329\,408w^{37}\\
& +997\,072\,075\,164\,663\,150\,542\,848w^{38}+789\,138\,181\,323\,007\,857\,786\,880w^{39}\\
& +388\,137\,877\,034\,203\,055\,390\,720w^{40}+4\,946\,627\,729\,914\,186\,432\,512w^{41}\\
& -26\,947\,297\,377\,570\,617\,556\,992w^{42}+10\,614\,515\,947\,351\,012\,540\,416w^{43}\\
& +998\,718\,253\,365\,681\,192\,960w^{44})
\end{aligned}
$$

$$
\begin{aligned}
p_0 = {} & 2w\cdot(-348+2768w+248\,784w^2-358\,217w^3-50\,461\,860w^4\\
& +16\,394\,998w^5+5\,283\,255\,372w^6+3\,911\,764\,831w^7-329\,364\,073\,508w^8\\
& -572\,985\,025\,996w^9+13\,847\,002\,317\,264w^{10}+38\,091\,073\,842\,520w^{11}\\
& -437\,846\,238\,222\,272w^{12}-1\,682\,624\,909\,395\,232w^{13}\\
& +10\,892\,230\,218\,721\,408w^{14}+52\,959\,188\,332\,189\,824w^{15}\\
& -214\,291\,413\,015\,639\,808w^{16}-1\,200\,734\,422\,407\,578\,112w^{17}\\
& +3\,319\,489\,124\,092\,462\,080w^{18}+20\,066\,023\,020\,568\,346\,624w^{19}\\
& -38\,248\,948\,302\,383\,529\,984w^{20}-254\,480\,826\,931\,185\,762\,304w^{21}\\
& +261\,281\,404\,771\,497\,082\,880w^{22}+2\,480\,194\,764\,802\,183\,397\,376w^{23}\\
& +148\,352\,203\,759\,030\,894\,592w^{24}-19\,049\,822\,668\,612\,433\,870\,848w^{25}\\
& -29\,328\,532\,357\,149\,024\,583\,680w^{26}+103\,410\,036\,785\,394\,615\,320\,576w^{27}\\
& +391\,034\,390\,334\,579\,595\,542\,528w^{28}+11\,096\,790\,708\,133\,489\,016\,832w^{29}\\
& -1\,530\,120\,948\,962\,096\,058\,466\,304w^{30}-2\,868\,669\,407\,093\,825\,701\,150\,720w^{31}\\
& -6\,126\,661\,019\,209\,831\,555\,268\,608w^{32}+2\,808\,943\,911\,875\,675\,603\,075\,072w^{33}\\
& +40\,458\,568\,379\,798\,955\,017\,371\,648w^{34}-169\,712\,327\,643\,359\,793\,079\,386\,112w^{35}\\
& -1092\,943\,871171\,162\,347\,998\,806\,016w^{36}-1781\,375\,524\,629\,107\,822\,238\,367\,744w^{37}\\
& +250\,471\,471\,742\,289\,487\,729\,786\,880w^{38}+4\,679\,788\,548\,889\,591\,917\,580\,386\,304w^{39}\\
& +7\,101\,176\,295\,364\,126\,941\,625\,974\,784w^{40}+5\,918\,768\,536\,906\,007\,398\,653\,624\,320w^{41}\\
& +4\,083\,406\,571\,846\,803\,705\,271\,681\,024w^{42}+2\,567\,747\,434\,748\,530\,216\,944\,009\,216w^{43}
\end{aligned}
$$

$$+ 846\,246\,487\,598\,480\,459\,424\,595\,968w^{44} - 49\,595\,159\,800\,068\,478\,383\,161\,344w^{45}$$
$$- 37\,040\,268\,890\,013\,610\,134\,208\,512w^{46} + 21\,784\,239\,691\,989\,525\,951\,676\,416w^{47}$$
$$+ 1\,753\,178\,556\,765\,355\,785\,584\,640w^{48}). \tag{A.4}$$

Appendix B. Solutions of the differential operator Y_3

Considering the critical exponents at the regular singular points, as well as the formal solutions of differential operator Y_3, one can make the following remarks. The roots of the polynomial of degree 28 in polynomial p_3 (see (A.4)) are *apparent singularities*. The roots of the polynomial of degree 4 in one of the factors of the same polynomial p_3 are not apparent singularities. While the formal solutions near $w = 0$, $w = \pm 1/4$ and $w = \infty$ have one Frobenius solution and two logarithmic solutions, the formal solutions near the other regular singular points are free of logarithmic solutions. The critical exponents at $w = 1, w = -1/2$, roots of $1 + 3w + 4w^2 = 0$, and roots of $1 - 3w - 18w^2 + 104w^3 + 96w^4 = 0$, are respectively $(-1, 0, 1)$, $(-1, 0, 1)$, $(-1, 0, 1)$ and $(-1, 1, 2)$. This leads us to look for the solutions of the third-order differential operator Y_3 as a linear combination of powers of elliptic integrals with a common factor 'taking care' of the non-logarithmic singularity behaviour of the singular points.

Defining

$$K(x) =_2 F_1(1/2, 1/2; 1; x), \qquad E(x) =_2 F_1(1/2, -1/2; 1; x)$$

and

$$s(w) = w^2 \cdot (1 - 16w^2)^3(1 + 2w)(1 - w)(1 + 3w + 4w^2)(1 - 3w - 18w^2 + 104w^3 + 96w^4)$$

one obtains the three independent solutions of the differential operator Y_3 as

$$S_1(Y_3) = \frac{1}{s(w)} \cdot (P_1 \cdot K^2(16w^2) + P_2 \cdot E^2(16w^2) + P_3 \cdot K(16w^2)E(16w^2))$$
$$S_2(Y_3) = \frac{1}{s(w)} \cdot (P_4 \cdot K^2(1/16w^2) - 16w^2 P_2 \cdot E^2(1/16w^2)$$
$$+ P_5 \cdot K(1/16w^2)E(1/16w^2))$$
$$S_3(Y_3) = \frac{1}{s(w)} \cdot ((P_1 + P_2 + P_3) \cdot K^2(1 - 16w^2) + P_2 \cdot E^2(1 - 16w^2)$$
$$- (2P_2 + P_3) \cdot K(1 - 16w^2)E(1 - 16w^2))$$

with

$$P_4 = -\frac{P_1}{16w^2} - \frac{(1 - 16w^2)^2}{16w^2} \cdot P_2 - \frac{1 - 16w^2}{16w^2} \cdot P_3,$$
$$P_5 = -2(1 - 16w^2) \cdot P_2 - P_3$$

where the three polynomials P_1, P_2 and P_3 read

$$P_1 = -(1 + 4w)(1 - 5w - 69w^2 + 537w^3 + 2964w^4 - 4100w^5$$
$$- 46\,816w^6 - 74\,688w^7 + 230\,656w^8 + 647\,680w^9 + 475\,136w^{10}$$
$$- 8192w^{11} + 720\,896w^{12})$$
$$P_2 = -1 + 5w + 25w^2 - 9w^3 - 2408w^4 - 17\,460w^5 - 19\,696w^6$$
$$+ 28\,800w^7 - 3328w^8 - 62\,464w^9 - 36\,864w^{10}$$
$$P_3 = 2 \cdot (1 - 3w - 65w^2 + 143w^3 + 3888w^4 + 15\,144w^5 - 10\,624w^6 - 172\,416w^7$$
$$- 241\,536w^8 + 111\,616w^9 + 282\,624w^{10} + 180\,224w^{11} + 98\,304w^{12}).$$

Remark. Let us note the very close similarity between the differential operator Y_3, occurring at the left of differential operator L_6 (see (7)) for $\tilde{\chi}^{(3)}$, and the differential operator M_2 (see (42)) occurring at the left of differential operator $\mathcal{L}_8$ for $\tilde{\chi}^{(4)}$. For this order 4 differential operator M_2, we have been able, using the same ansatz, to obtain in the closed form three of the four solutions, also expressed as a linear combination of products of elliptic integrals. Note that, setting $\lambda = 16w^2$, one can detect in the solutions of Y_3 (and also in the three solutions of M_2 we have found) the structure of Σ_3 permutation group [29], λ, $1/\lambda$, $1-\lambda$, $1-1/\lambda$, etc.

Appendix C. Connection matrices between $w = 0$ and $w = -1/4$, $w = \infty$

C.1. The connection matrix between $w = 0$ and $w = -1/4$

The basis of solutions at the anti-ferromagnetic critical point $w = -1/4$ are chosen as follows (with $x = 1 + 4w$),

$$
\begin{aligned}
S_1^{(-1/4)}(x) &= \mathcal{S}(N_1)(x),\\
S_2^{(-1/4)}(x) &= [1, 0, 1/10, -87/700, -313/1680, \ldots],\\
S_3^{(-1/4)}(x) &= [0, 1, -17/10, -23/25, -1/30, \ldots],\\
S_4^{(-1/4)}(x) &= [1, -5/2, -3/8, 5/16, 83/512, \ldots],\\
S_5^{(-1/4)}(x) &= S_4^{(-1/4)}(x)\cdot(\ln(x/8)+6) + S_{50}^{(-1/4)}(x)\\
S_6^{(-1/4)}(x) &= S_4^{(-1/4)}(x)\cdot(\ln^2(x/8) + 12\ln(x/8) + 23264/315)\\
&\quad + 2S_{50}^{(-1/4)}(x)\cdot(\ln(x/8)+6) + S_{60}^{(-1/4)}(x)
\end{aligned}
$$

with

$$
\begin{aligned}
S_{50}^{(-1/4)}(x) &= [0, 97/6, 553/240, -2339/672, -1\,678\,457/645\,120, \ldots],\\
S_{60}^{(-1/4)}(x) &= [0, 0, 0, 85\,997/18\,000, 8\,450\,503/1\,814\,400, \ldots].
\end{aligned}
$$

Here again, an optimal choice of the components is made in order to remove logarithms and to have as many zeros as possible in the entries of the matrix. The same method of matching the series solutions at a half-way point between $w = 0$ and $w = -1/4$, gives

$$
C(0,-1/4) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
-2 & r_{22} & r_{23} & 0 & 0 & 0\\
-2\pi\mathrm{i} & r_{32}+r_{22}\pi\mathrm{i} & r_{33}+r_{23}\pi\mathrm{i} & 0 & 0 & 0\\
6 & \dfrac{1}{\pi}i_{52} & \dfrac{1}{\pi}i_{53} & 0 & 0 & \dfrac{1}{8\pi^2}\\
\dfrac{5}{2}+6\pi\mathrm{i} & a_{52} & a_{53} & 0 & \dfrac{1}{16} & \dfrac{1}{8\pi}\mathrm{i}\\
-\dfrac{23}{8}-\dfrac{17\pi^2}{3}+5\pi\mathrm{i} & a_{62} & a_{63} & \dfrac{\pi^2}{32} & \dfrac{\pi}{8}\mathrm{i} & -\dfrac{1}{8}
\end{bmatrix} \qquad \text{(C.1)}
$$

with

$$
\begin{aligned}
&r_{22}r_{33} - r_{23}r_{32} = 25/12288\\
&a_{52} = -3r_{32} - \frac{5}{4}r_{22} + i_{52}\mathrm{i}, \qquad a_{53} = -3r_{33} - \frac{5}{4}r_{23} + i_{53}\mathrm{i}\\
&a_{62} = \left(\frac{25}{16} - \frac{2\pi^2}{3} - \frac{5\pi}{2}\mathrm{i}\right)r_{22} - \left(\frac{5}{2}+6\pi\mathrm{i}\right)r_{32} - i_{52}\pi
\end{aligned}
$$

$$a_{63} = \left(\frac{25}{16} - \frac{2\pi^2}{3} - \frac{5\pi}{2}\mathrm{i}\right) r_{23} - \left(\frac{5}{2} + 6\pi\mathrm{i}\right) r_{33} - i_{53}\pi$$

and where[17]

$$\begin{aligned} &r_{22} \simeq -0.059\,050\,961\,331, && r_{23} \simeq -0.086\,431\,902\,55,\\ &r_{32} \simeq 0.163\,138\,242\,313\,1, && i_{52} \simeq -1.839\,621\,665\,835,\\ &i_{53} \simeq -0.015\,467\,563\,102. \end{aligned}$$

C.2. The connection matrix between $w = 0$ and $w = \infty$

The bases of solutions at the singular point $w = \infty$ are chosen as follows (with $x = 1/w$),

$$\begin{aligned} &S_1^{(\infty)}(x) = \mathcal{S}(N_1),\\ &S_2^{(\infty)}(x) = [1, 1, 7/16, 1/16, 7/256, \ldots],\\ &S_3^{(\infty)}(x) = (\ln(x/4) - 2/3)\cdot S_2^{(\infty)}(x) + S_{30}^{(\infty)}(x),\\ &S_4^{(\infty)}(x) = [0, 1, 0, 1/32, -9/512, \ldots],\\ &S_5^{(\infty)}(x) = (\ln(x/16) + a_1)\cdot S_4^{(\infty)}(x) + S_{50}^{(\infty)}(x),\\ &S_6^{(\infty)}(x) = (\ln^2(x/16) + 2a_1\ln(x/16) + a_2)\cdot S_4^{(\infty)}(x) + 2(\ln(x/16) + a_1)\cdot S_{50}^{(\infty)}(x) + S_{60}^{(\infty)}(x) \end{aligned} \tag{C.2}$$

with

$$\begin{aligned} &a_1 = -5 - \frac{\pi}{2}\mathrm{i}, \qquad a_2 = -\frac{\pi^2}{4} + \frac{379}{11} + 5\pi\mathrm{i},\\ &S_{30}^{(\infty)}(x) = [2/3, 1/6, 1/24, -1/96, 7/768, \ldots],\\ &S_{50}^{(\infty)}(x) = [0, 0, -3/2, 3/64, -107/512, -23113/491520, \ldots],\\ &S_{60}^{(\infty)}(x) = [0, 0, 0, 93/44, -80891/13516, 105811/4055040, \ldots]. \end{aligned}$$

The connection matrix reads

$$C(0,\infty) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 1 & -\frac{1}{16} & -\frac{3}{16\pi}\mathrm{i} & 0 & 0 & 0\\ -\pi\mathrm{i} & 0 & -\frac{1}{16} & 0 & 0 & 0\\ -11 + y_{41}\mathrm{i} & x_{42} - \frac{1}{\pi}\mathrm{i} & \frac{2}{\pi^2} - \frac{15}{16\pi}\mathrm{i} & 0 & 0 & \frac{1}{4\pi^2}\\ a_{51} & a_{52} & -\frac{9}{16} - \frac{49}{64\pi}\mathrm{i} & 0 & \frac{1}{16} & -\frac{1}{8\pi}\mathrm{i}\\ a_{61} & a_{62} & -\frac{11}{32} + \frac{5\pi}{16}\mathrm{i} - \frac{75}{256\pi}\mathrm{i} & \frac{\pi^2}{64} & -\frac{\pi}{16}\mathrm{i} & -\frac{1}{16} \end{bmatrix} \tag{C.3}$$

where

$$\begin{aligned} &x_{42} \simeq -1.534\,248\,223\,197, && y_{41} \simeq -22.932\,479\,960\,454,\\ &a_{51} = -\frac{5}{4} + \frac{\pi}{2}y_{41} + 7\pi\mathrm{i}, && a_{52} = -\frac{11}{64} - \frac{\pi}{2}x_{42}\mathrm{i} - \frac{\pi}{32}\mathrm{i},\\ &a_{61} = \frac{29}{16} + \frac{16\pi^2}{3} - \frac{\pi^2}{4}\mathrm{i}y_{41} + \frac{5\pi}{2}\mathrm{i}, && a_{62} = -\frac{25}{256} - \frac{7\pi^2}{192} - \frac{\pi^2}{4}x_{42}. \end{aligned}$$

[17] The numbers r_{ij} and i_{ij} are peculiar to each connection matrix.

Appendix D

D.1. Basis of solutions for $w = 1$, $w = -1/2$ *and* $1 + 3w + 4w^2 = 0$

The basis near $w = 1$ is (with $x = 1 - w$)

$$
\begin{aligned}
&S_1^{(1)}(x) = \mathcal{S}(N_1)(x),\\
&S_2^{(1)}(x) = [0, 0, 0, 1, 65/24, 383/72, \ldots],\\
&S_3^{(1)}(x) = S_2^{(1)}(x) \cdot (\ln(x/24) + 2666/75) + S_{30}^{(1)}(x),\\
&S_4^{(1)}(x) = [0, 1, 0, 0, 0, -213\,149\,176\,769/914\,630\,737\,500, \ldots],\\
&S_5^{(1)}(x) = [0, 0, 1, 0, 0, 806\,017\,240\,807/426\,827\,677\,500, \ldots],\\
&S_6^{(1)}(x) = [0, 0, 0, 0, 1, 555\,108\,887/158\,084\,325, \ldots],
\end{aligned}
$$

with

$$S_{30}^{(1)}(x) = [0, 96/5, 628/25, 0, -812\,657/18\,000, \ldots].$$

The basis near $w = -1/2$ reads (with $x = 1 + 2w$)

$$
\begin{aligned}
&S_1^{(-1/2)}(x) = \mathcal{S}(N_1)(x),\\
&S_2^{(-1/2)}(x) = [0, 0, 0, 1, 8/3, 46/9, 247/27, \ldots],\\
&S_3^{(-1/2)}(x) = S_2^{(-1/2)}(x) \cdot \ln(x) + S_{30}^{(-1/2)}(x),\\
&S_4^{(-1/2)}(x) = [0, 1, 0, 0, 0, -55\,489/60\,345, \ldots],\\
&S_5^{(-1/2)}(x) = [0, 0, 1, 0, 0, 159\,977/80\,460, \ldots],\\
&S_6^{(-1/2)}(x) = [0, 0, 0, 0, 1, 1492/447, \ldots]
\end{aligned}
$$

where

$$S_{30}^{(-1/2)}(x) = [0, 3/4, 7/8, 0, -95/144, \ldots].$$

The basis near $w_1 = -3/8 + i\sqrt{7}/8$ root of $1 + 3w + 4w^2$ is (with $x = 1 - w/w_1$)

$$
\begin{aligned}
&S_1^{(w_1)}(x) = \mathcal{S}(N_1)(x),\\
&S_2^{(w_1)}(x) = [0, 1, 49/64 - 61/(64\sqrt{7})\mathrm{i}, 655/1024 - 747/(1024\sqrt{7})\mathrm{i}, \ldots],\\
&S_3^{(w_1)}(x) = S_2^{(w_1)}(x) \cdot \ln(x) + S_{30}^{(w_1)}(x), \qquad S_4^{(w_1)}(x) = [0, 0, 1, 0, 0, \ldots],\\
&S_5^{(w_1)}(x) = [0, 0, 0, 1, 0, \ldots], \qquad S_6^{(w_1)}(x) = [0, 0, 0, 0, 1, \ldots]
\end{aligned}
$$

with

$$S_{30}^{(w_1)}(x) = [0, 0, 657/896 + 61/(128\sqrt{7})\mathrm{i}, 41\,203/43\,008 + 1991/(6144\sqrt{7})\mathrm{i}, \ldots].$$

D.2. Connection matrices for $w = 1$, $w = -1/2$ *and* $1 + 3w + 4w^2 = 0$

For the singular point $w = 1$, the connection matrix with $w = 0$ reads

$$C(0, 1) = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{bmatrix}$$

where $\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix}$ and $[\mathbf{C}]$ read

$$\begin{bmatrix}
1 & 0 & 0\\
4+i_{21}\mathrm{i} & -\dfrac{\sqrt{3}}{144}\mathrm{i} & -\dfrac{\sqrt{3}}{144\pi}\\
-2\pi\mathrm{i} & -\dfrac{\pi\sqrt{3}}{216} & 0\\
-4-\dfrac{4}{\pi}i_{51}-\dfrac{5}{\pi}i_{21}+i_{41}\mathrm{i} & r_{42}+\dfrac{4}{\pi}r_{52}\mathrm{i}+\dfrac{\sqrt{3}}{48}i & -\dfrac{7\sqrt{3}}{48\pi}\\
5-\dfrac{2}{\pi}i_{61}+\left(\dfrac{2\pi}{3}+\dfrac{25}{8\pi}\right)i_{21}+i_{51}\mathrm{i} & r_{52}+\dfrac{2}{\pi}r_{62}\mathrm{i}-\dfrac{25\sqrt{3}}{1728}\mathrm{i} & \dfrac{5\sqrt{3}}{576\pi}+\dfrac{\sqrt{3}}{18}\mathrm{i}\\
\dfrac{13}{2}+\dfrac{\pi^2}{3}+i_{61}\mathrm{i} & r_{62}-\dfrac{\pi^2\sqrt{3}}{432}\mathrm{i}-\dfrac{25\sqrt{3}}{2304}\mathrm{i} & \dfrac{11\pi\sqrt{3}}{432}-\dfrac{25\sqrt{3}}{2304\pi}
\end{bmatrix}$$

$$\begin{bmatrix}
r_{44}+i_{44}\mathrm{i} & r_{45}+i_{45}\mathrm{i} & r_{46}+i_{46}\mathrm{i}\\
\dfrac{\pi}{4}i_{44}+i_{54}\mathrm{i} & \dfrac{\pi}{4}i_{45}+i_{55}\mathrm{i} & \dfrac{\pi}{4}i_{46}+i_{56}\mathrm{i}\\
\dfrac{\pi}{2}i_{54} & \dfrac{\pi}{2}i_{55} & \dfrac{\pi}{2}i_{56}
\end{bmatrix}$$

where

$i_{21} \simeq 1.838\,093\,775\,180$, $i_{41} \simeq 4.136\,525\,226\,980$, $i_{51} \simeq -8.138\,989\,276\,03$

$i_{61} \simeq 20.743\,660\,887\,04$, $r_{42} \simeq 2.542\,631\,644\,752$, $r_{52} \simeq -0.011\,842\,088\,97$

$r_{62} \simeq -4.871\,087\,773\,44$, $r_{44} \simeq 1.622\,875\,171\,987$, $r_{45} \simeq 1.954\,781\,507\,112$

$r_{46} \simeq -3.513\,874\,999\,53$, $i_{44} \simeq 0.158\,271\,118\,920$, $i_{54} \simeq -2.138\,739\,670\,59$

$i_{45} \simeq 0.041\,310\,289\,307$, $i_{55} \simeq -2.467\,598\,547\,30$, $i_{46} \simeq -0.028\,730\,643\,96$

$i_{56} \simeq 4.392\,293\,882\,282$,

These numbers are such that

$$i_{46}i_{55}r_{44}+r_{45}i_{56}i_{44}-r_{46}i_{55}i_{44}-i_{46}r_{45}i_{54}+i_{45}r_{46}i_{54}-i_{45}i_{56}r_{44}=-\frac{468\,398}{18\,984\,375\pi^2}.$$

The connection matrix between $w=0$ and the singular point $w=-1/2$, reads

$$C(0,-1/2)=\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{B} & \mathbf{C}\end{bmatrix}$$

where $\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix}$ and $[\mathbf{C}]$ read

$$\begin{bmatrix}
1 & 0 & 0\\
-2-\dfrac{3\ln(2)}{\pi}\mathrm{i} & r_{22} & -\dfrac{\sqrt{3}}{9\pi}\\
-\pi\mathrm{i}+3\ln(2) & -\dfrac{\pi\sqrt{3}}{54}+\pi r_{22}\mathrm{i} & \dfrac{-\sqrt{3}}{9}\mathrm{i}\\
r_{41}+i_{41}\mathrm{i} & r_{42}+i_{42}\mathrm{i} & \dfrac{11\sqrt{3}}{9\pi}\\
r_{51}+i_{51}\mathrm{i} & r_{52}+i_{52}\mathrm{i} & \dfrac{5\sqrt{3}}{36\pi}+\dfrac{7\sqrt{3}}{9}\mathrm{i}\\
r_{61}+i_{61}\mathrm{i} & r_{62}+i_{62}\mathrm{i} & -\dfrac{25\sqrt{3}}{144\pi}-\dfrac{13\pi\sqrt{3}}{27}+\dfrac{5\sqrt{3}}{18}\mathrm{i}
\end{bmatrix},$$

$$\begin{bmatrix} r_{44}+i_{44}\mathrm{i} & r_{45}+i_{45}\mathrm{i} & r_{46}+i_{46}\mathrm{i} \\ -\dfrac{3\pi}{4}i_{44}+i_{54}\mathrm{i} & -\dfrac{3\pi}{4}i_{45}+i_{55}\mathrm{i} & -\dfrac{3\pi}{4}i_{46}+i_{56}\mathrm{i} \\ r_{64}-\dfrac{\pi^2}{2}i_{44}\mathrm{i} & r_{65}-\dfrac{\pi^2}{2}i_{45}\mathrm{i} & r_{66}-\dfrac{\pi^2}{2}i_{46}\mathrm{i} \end{bmatrix}$$

where

$r_{22} \simeq -0.025\,399\,597\,75$, $r_{41} \simeq 6.805\,351\,589\,429$, $r_{51} \simeq 7.203\,810\,787\,172$,
$r_{61} \simeq -8.757\,986\,516\,23$, $i_{41} \simeq -5.235\,292\,153\,52$, $i_{51} \simeq 12.149\,726\,439\,02$,
$i_{61} \simeq 7.505\,979\,318\,469$, $r_{42} \simeq 0.512\,271\,205\,543$, $r_{52} \simeq -0.754\,975\,549\,89$,
$r_{62} = 2.232\,400\,538\,972$, $i_{42} \simeq 0.462\,196\,540\,081$, $i_{52} \simeq 0.143\,220\,115\,658$,
$i_{62} \simeq -0.181\,954\,276\,23$, $r_{44} \simeq -0.168\,129\,055\,3$, $r_{45} \simeq -0.002\,706\,580\,55$,
$r_{46} \simeq -0.003\,230\,432\,90$, $i_{44} \simeq -0.143\,012\,924\,13$, $i_{45} \simeq 0.690\,508\,507\,395$,
$i_{46} \simeq -1.263\,549\,266\,77$, $i_{54} \simeq -0.348\,445\,547\,01$, $r_{64} \simeq 0.812\,327\,323\,812$,
$i_{55} \simeq -0.501\,086\,485\,04$, $r_{65} \simeq 2.347\,957\,990\,666$, $i_{56} \simeq 1.132\,041\,888\,142$,
$r_{66} \simeq -5.350\,563\,266\,40$.

The connection matrix between $w = 0$ and the singular point $w_1 = -3/8 + i\sqrt{7}/8$ root of $1 + 3w + 4w^2 = 0$, reads

$$C(0, w_1) = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{bmatrix}$$

where $\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}$ and $[\mathbf{C}]$ read

$$\begin{bmatrix} 1 & 0 & 0 \\ r_{21}-\dfrac{3}{2\pi}r_{31}\mathrm{i} & r_{22}-\dfrac{3}{2\pi}r_{32}\mathrm{i}+\dfrac{275\sqrt{7}}{16384}\mathrm{i} & a \\ r_{31}+\dfrac{2\pi}{3}r_{21}\mathrm{i}+\dfrac{2\pi}{3}\mathrm{i} & r_{32}+\dfrac{2\pi}{3}r_{22}\mathrm{i}-\dfrac{623\pi}{24576}\mathrm{i} & \dfrac{2\pi}{3}\mathrm{i}a \\ r_{41}+i_{41}\mathrm{i} & r_{42}+i_{42}\mathrm{i} & -\dfrac{1}{3}a \\ r_{51}+i_{51}\mathrm{i} & r_{52}+i_{52}\mathrm{i} & \left(-\dfrac{5}{4}+\dfrac{2\pi}{3}\mathrm{i}\right)a \\ r_{61}+i_{61}\mathrm{i} & r_{62}+i_{62}\mathrm{i} & \left(\dfrac{25}{16}-\pi^2-\dfrac{5\pi}{3}\mathrm{i}\right)a \end{bmatrix}$$

$$\begin{bmatrix} r_{44}+i_{44}\mathrm{i} & r_{45}+i_{45}\mathrm{i} & r_{46}+i_{46}\mathrm{i} \\ r_{54}+i_{54}\mathrm{i} & r_{55}+i_{55}\mathrm{i} & r_{56}+i_{56}\mathrm{i} \\ r_{64}+i_{64}\mathrm{i} & r_{65}+i_{65}\mathrm{i} & r_{66}+i_{66}\mathrm{i} \end{bmatrix}$$

where

$$a = \frac{825\sqrt{7}-1869\mathrm{i}}{16\,384\pi},$$

$r_{21} \simeq -0.309\,839\,631\,51$, $r_{31} \simeq 1.386\,294\,361\,11$, $r_{22} \simeq -0.079\,967\,467\,93$,
$r_{32} \simeq 0.044\,743\,829\,620$, $r_{41} \simeq 4.703\,166\,105\,99$, $i_{41} \simeq -5.102\,032\,209\,92$,
$r_{42} \simeq 0.028\,522\,637\,766$, $i_{42} \simeq 0.037\,312\,675\,44$, $r_{51} \simeq 1.404\,170\,417\,754$,
$i_{51} \simeq 10.771\,852\,695\,95$, $r_{52} \simeq 0.256\,542\,990\,02$, $i_{52} \simeq -0.036\,953\,282\,52$,
$r_{61} \simeq -6.988\,982\,509\,54$, $i_{61} \simeq -17.585\,497\,074$, $r_{62} \simeq -0.183\,427\,057\,50$,
$i_{62} \simeq 1.339\,914\,984\,659$, $r_{44} \simeq 0.003\,948\,320\,42$, $i_{44} \simeq 0.043\,931\,830\,095$,
$r_{45} \simeq -0.027\,162\,803\,32$, $i_{45} \simeq -0.090\,075\,389\,9$, $r_{46} \simeq 0.070\,134\,204\,478$,
$i_{46} \simeq 0.050\,869\,745\,772$, $r_{54} \simeq -0.212\,294\,769\,9$, $i_{54} \simeq 0.033\,562\,029\,788$,
$r_{55} \simeq 0.496\,361\,798\,471$, $i_{55} \simeq 0.004\,559\,664\,93$, $r_{56} \simeq -0.368\,676\,471\,37$,
$i_{56} \simeq 0.040\,697\,038\,977$, $r_{64} \simeq -0.127\,940\,761\,2$, $i_{64} \simeq -0.683\,828\,600\,60$,
$r_{65} \simeq -0.147\,391\,270\,07$, $i_{65} \simeq 1.645\,961\,232\,66$, $r_{66} \simeq 0.189\,914\,623\,980$,
$i_{66} \simeq -1.294\,833\,256\,56$.

Appendix E. Monodromy matrices in the $w = 0$ basis

The monodromy matrix around $w = 0$ expressed in terms of its own ($w = 0$) well-suited basis is given in (33).

The monodromy matrix around $w = -1/2$, expressed in terms of the ($w = 0$) well-suited basis, after a conjugation similar to (34), and thus using the previously given connection matrices, reads in terms of α and Ω,

$$4\alpha^2 \cdot M_{w=0}(-1/2)(\alpha, \Omega) = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{bmatrix}$$

where

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} 4\alpha^2 & 0 & 0 \\ 48\alpha\Omega & 4\alpha\,(12\Omega + \alpha) & -96\Omega \\ 24\Omega\alpha^2 & 24\Omega\alpha^2 & 4\,(\alpha - 12\Omega)\,\alpha \\ -528\alpha\Omega & -528\alpha\Omega & 1056\Omega \\ -12\,(14\alpha + 5)\,\alpha\Omega & -12\,(14\alpha + 5)\,\alpha\Omega & 24\,(14\alpha + 5)\,\Omega \\ -\alpha a\Omega & -\alpha a\Omega & 2\Omega a \end{bmatrix}$$

with $a = (-75 + 52\alpha^2 + 60\alpha)$ and $[\mathbf{C}] = 4\alpha^2 \cdot \mathbf{Id}(\mathbf{3} \times \mathbf{3})$.

The monodromy matrix around $w = 1/4$, expressed in terms of the ($w = 0$) well-suited basis reads

$$24\alpha^4 \cdot M_{w=0}(1/4)(\alpha, \Omega) = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{bmatrix}$$

where $\left[\begin{smallmatrix} \mathbf{A} \\ \mathbf{B} \end{smallmatrix}\right]$ and $[\mathbf{C}]$ read, respectively,

$$\begin{bmatrix} -24\alpha^4 & 0 & 0 \\ -48\alpha^4 & 24\alpha^4 & -144\alpha^2\Omega \\ 0 & 0 & 24\alpha^4 \\ -48(5\alpha^4 + 8\Omega^2 + 8\Omega^2\alpha^2) & 32(4\Omega\alpha^2 - 75\Omega - 15\alpha^2)\Omega & 48(9\alpha^2 + 80\Omega)\Omega \\ 12(5\alpha^2 + 4\Omega + 4\Omega\alpha^2)\alpha^2 & 4(75 - 4\alpha^2)\alpha^2\Omega & -300\alpha^2\Omega \\ -(87 + 8\alpha^2)\alpha^4 & 0 & 3(4\alpha^2 - 75)\alpha^2\Omega \end{bmatrix},$$

and

$$\begin{bmatrix} 24\alpha^4 & -384\alpha^2\Omega & 1536\Omega^2 \\ 0 & 24\alpha^4 & -192\alpha^2\Omega \\ 0 & 0 & 24\alpha^4 \end{bmatrix}.$$

The monodromy matrix around $w = -1/4$, expressed in terms of the $(w = 0)$ well-suited basis reads

$$12\alpha^4 \cdot M_{w=0}(-1/4)(\alpha, \Omega) = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{bmatrix}$$

where $\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}$ and $[\mathbf{C}]$ read, respectively,

$$\begin{bmatrix} -12\alpha^4 & 0 & 0 \\ 48\alpha^4 & 12\alpha^4 & 0 \\ 24\alpha^5 & 0 & 12\alpha^4 \\ a_{41} & a_{42} & 192\Omega(10\Omega - 3\alpha^2) \\ a_{51} & a_{52} & 48\alpha\Omega(-5\alpha + 20\Omega - 6\alpha^2) \\ a_{61} & a_{62} & 48\alpha^2\Omega(-5\alpha + 10\Omega - 3\alpha^2) \end{bmatrix},$$

with

$$\begin{aligned} a_{41} &= -144\alpha^4 - 192\Omega^2 - 192\Omega^2\alpha^2, \\ a_{42} &= -16\Omega(60\alpha\Omega + 75\Omega + 8\Omega\alpha^2 - 18\alpha^3 + 15\alpha^2), \\ a_{51} &= -12\alpha(5\alpha^3 + 6\alpha^4 + 8\Omega^2 + 8\Omega^2\alpha^2 - 2\alpha\Omega - 2\alpha^3\Omega), \\ a_{52} &= -2\alpha\Omega(300\Omega + 32\Omega\alpha^2 + 240\alpha\Omega - 80\alpha^3 - 75\alpha), \\ a_{61} &= -\alpha^2(-69\alpha^2 + 60\alpha^3 + 34\alpha^4 + 48\Omega^2 + 48\Omega^2\alpha^2 - 24\alpha\Omega - 24\alpha^3\Omega), \\ a_{62} &= -2\alpha^2\Omega(150\Omega - 30\alpha^2 + 16\Omega\alpha^2 + 120\alpha\Omega - 44\alpha^3 - 75\alpha), \end{aligned}$$

and

$$\begin{bmatrix} 12(\alpha + 4\Omega)^2\alpha^2 & -192(\alpha + 4\Omega)\alpha\Omega & 768\Omega^2 \\ 24\alpha^3\Omega(\alpha + 4\Omega) & -12\alpha^2(-\alpha^2 + 32\Omega^2) & 96(-\alpha + 4\Omega)\alpha\Omega \\ 48\alpha^4\Omega^2 & 48(\alpha - 4\Omega)\alpha^3\Omega & 12(-\alpha + 4\Omega)^2\alpha^2 \end{bmatrix}.$$

The monodromy matrix around $w = \infty$, expressed in terms of the $w = 0$ well-suited basis reads

$$24\alpha^4 \cdot M_{w=0}(\infty)(\alpha, \Omega) = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{bmatrix}$$

where $\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}$ and $[\mathbf{C}]$ read, respectively,

$$\begin{bmatrix} 24\alpha^4 & 0 & 0 \\ -288\alpha^3\Omega & -24\alpha^3(-\alpha + 6\Omega) & -864\Omega\alpha^2 \\ 48\alpha^4\Omega & 24\alpha^4\Omega & 24\alpha^3(\alpha + 6\Omega) \\ a_{41} & a_{42} & 96(-21\alpha^2 + 160\Omega)\Omega \\ a_{51} & a_{52} & -120(-6\alpha^2 - \alpha + 32\Omega)\alpha\Omega \\ a_{61} & a_{62} & 6(20\alpha - 225 - 36\alpha^2 + 160\Omega)\alpha^2\Omega \end{bmatrix}$$

with

$$a_{41} = 96(\alpha^3 + 16\Omega\alpha^2 - 16\Omega)\Omega$$

$$
\begin{aligned}
a_{42} &= 16(-33\alpha^3 - 60\alpha^2 + 240\alpha\Omega + 8\Omega\alpha^2 - 600\Omega)\Omega,\\
a_{51} &= -24(2\alpha^3 - 15\alpha^2 - 4\alpha + 16\Omega\alpha^2 - 16\Omega)\alpha\Omega,\\
a_{52} &= -4(-40\alpha^3 - 45\alpha^2 - 150\alpha + 240\alpha\Omega + 8\Omega\alpha^2 - 600\Omega)\alpha\Omega,\\
a_{61} &= 6(-20\alpha^2 - 83\alpha + 4\alpha^3 + 16\Omega\alpha^2 - 16\Omega)\alpha^2\Omega,\\
a_{62} &= (-525\alpha - 44\alpha^3 + 240\alpha\Omega + 8\Omega\alpha^2 - 600\Omega)\alpha^2\Omega,
\end{aligned}
$$

and

$$
\begin{bmatrix}
24(-\alpha+4\Omega)^2\alpha^2 & 768\Omega(-\alpha+4\Omega)\alpha & 6144\Omega^2\\
-24\alpha^3\Omega(-\alpha+4\Omega) & -24\alpha^2(-\alpha^2+32\Omega^2) & -384\alpha\Omega(\alpha+4\Omega)\\
24\alpha^4\Omega^2 & 48\alpha^3\Omega(\alpha+4\Omega) & 24\alpha^2(\alpha+4\Omega)^2
\end{bmatrix}.
$$

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