Ramanujan J (2007) 13:63–101
DOI 10.1007/s11139-006-0243-3

On the dynamics of certain recurrence relations

D. Borwein · J. Borwein · R. Crandall · R. Mayer

Dedicated respectfully to Richard Askey on the occasion of his 70th Birthday.
Received: 15 January 2004 / Accepted: 15 July 2004

Abstract In recent analyses [3, 4] the remarkable AGM continued fraction of Ramanujan—denoted $\mathcal{R}_1(a, b)$—was proven to converge for almost all complex parameter pairs (a, b). It was conjectured that $\mathcal{R}_1$ diverges if and only if $(0 \neq a = be^{i\phi}$ with $\cos^2\phi \neq 1)$ or $(a^2 = b^2 \in (-\infty, 0))$. In the present treatment we resolve this conjecture to the positive, thus establishing the precise convergence domain for $\mathcal{R}_1$. This is accomplished by analyzing, using various special functions, the dynamics of sequences such as (t_n) satisfying a recurrence

$$t_n = (t_{n-1} + (n-1)\kappa_{n-1}t_{n-2})/n,$$

where $\kappa_n := a^2, b^2$ as n be even, odd respectively.

Research supported by NSERC.

Research supported by NSERC, the Canada Foundation for Innovation and the Canada Research Chair Program.

D. Borwein (✉)
Department of Mathematics, University of Western Ontario, London Ontario Canada N6A 5B7
e-mail: dborwein@uwo. ca

J. Borwein
Faculty of Computer Science Dalhousie University, Halifax NS Canada B3H 1W5
e-mail: jborwein@cs.dal.ca

R. Crandall
Center for Advanced Computation, Reed College, Portland, Oregon 97202
e-mail: crandall@reed.edu

R. Mayer
Department of Mathematics, Reed College, Portland Oregon 97202
e-mail: mayer@reed.edu

As a byproduct, we are able to give, in some cases, exact expressions for the n-th convergent to the fraction $\mathcal{R}_1$, thus establishing some precise convergence rates. It is of interest that this final resolution of convergence depends on rather intricate theorems for complex-matrix products, which theorems evidently being extensible to more general continued fractions.

Keywords Complex continued fractions · Dynamical systems · Arithmetic-geometric mean · Matrix analysis · Stability theory

2000 Mathematics Subject Classification Primary—11J70, 11Y65, 40A15

1 Nomenclature

In companion treatments [3, 4] we considered the Ramanujan AGM fraction

$$\mathcal{R}_1(a,b) = \frac{a}{1+\mathcal{S}(a,b)} \frac{a}{1+\cfrac{b^2}{1+\cfrac{4a^2}{1+\cfrac{9b^2}{1+\ddots}}}} \tag{1.1}$$

one of whose attractive properties being a formal AGM relation—known to be true at least for positive real a, b—

$$\mathcal{R}_1\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{\mathcal{R}_1(a,b)+\mathcal{R}_1(b,a)}{2},$$

but of dubious validity for general complex parameters [3]. The work [4] focused on the convergence domain

$$\mathcal{D}_0 := \{(a,b) \in \mathcal{C} \times \mathcal{C} : \mathcal{R}_1(a,b) \text{ converges on } \hat{\mathcal{C}}\},$$

where $\hat{\mathcal{C}} := \mathcal{C} \cup \{\infty\}$ denotes the extended complex field. It was proved therein that if we define

$$\begin{aligned}
\mathcal{D}_2 &:= \{(a,b) \in \mathcal{C} \times \mathcal{C} : |a| \neq |b|\},\\
\mathcal{D}_3 &:= \{(a,b) \in \mathcal{C} \times \mathcal{C} : a^2 = b^2 \notin (-\infty, 0)\},\\
\mathcal{D}_1 &:= \mathcal{D}_2 \cup \mathcal{D}_3,
\end{aligned}$$

then

$$\mathcal{D}_1 \subseteq \mathcal{D}_0,$$

so that the Ramanujan fraction converges for almost all complex pairs (a, b). There is a conjecture [4, Conjecture 5.4], effectively saying that in fact

$$\mathcal{D}_1 = \mathcal{D}_0.$$

Equivalently: The fraction $\mathcal{R}_1$ diverges whenever ($0 \neq a = be^{i\phi}$ with $\cos^2\phi \neq 1$) or ($a^2 = b^2 \in (-\infty, 0)$). This divergence behavior can be understood, as we do presently, in terms of the special dynamics of certain recurrence relations.

In what follows, we consider classical convergents p_n/q_n to the fraction $\mathcal{S}$ from (1.1), but renormalize to obtain certain sequences. Specifically, to evaluate $\mathcal{S}$ we use initial values $(p_{-1}, p_0, q_{-1}, q_0) = (1, 0, 0, 1)$ and a recurrence (also satisfied by the p_n)

$$q_n = q_{n-1} + n^2\kappa_n q_{n-2},$$

where $\kappa_n = a^2, b^2$ as n be even, odd respectively. Whether this classical procedure has p_n/q_n approaching a limit depends in a delicate way, as we have intimated, on the parameters *a, b*.

For the theoretical treatments to follow, we now establish some sequence renormalizations, each theoretically interesting in its own right. We shall consider renormalized sequences denoted (t_n), (r_n), (v_n) in what follows. The first renormalization is

$$t_n := \frac{q_{n-1}}{n!},$$

so that

$$t_n = \frac{t_{n-1} + (n-1)\kappa_{n-1}t_{n-2}}{n}, \tag{1.2}$$

as intimated in our Abstract. Another interesting renormalization is

$$r_n := \frac{q_n}{a^n\Gamma(n+3/2)}, \tag{1.3}$$

with recurrence

$$r_n = \frac{1}{a(n+1/2)}r_{n-1} + \frac{n^2}{n^2-1/4}r_{n-2}, \quad n \text{ even},$$
$$r_n = \frac{1}{a(n+1/2)}r_{n-1} + \frac{n^2e^{-2i\phi}}{n^2-1/4}r_{n-2}, \quad n \text{ odd},$$

as is used in Conjecture 5.4 of [4]. A slight variation on (r_n) turns out to be optimal in some ways:

$$v_n := \frac{q_n}{\Gamma(n+3/2)}\frac{1}{\kappa_n^{(n+1)/2}}. \tag{1.4}$$

The recurrence relation for this (v_n) sequence will later be put into a convenient matrix form. Now the classical separation of convergents to $\mathcal{S}$ can be written, for n even, as

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = -\frac{b^n a^n n!^2}{q_n q_{n-1}}.$$

Given our various renormalized sequences (t_n), (r_n), (v_n), we can also write (again, for n even)

$$\begin{aligned} \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} &= -\frac{b^n a^n}{t_{n+1} t_n (n+1)}, \\ &= -\frac{a(b/a)^n}{r_n r_{n-1}} \left\{1 + O\left(\frac{1}{n}\right)\right\}, \\ &= -\frac{1}{a v_n v_{n-1}} \left\{1 + O\left(\frac{1}{n}\right)\right\}. \end{aligned} \tag{1.5}$$

If we assume $a = be^{i\phi} \neq 0$, the $\mathcal{S}$ fraction—and hence $\mathcal{R}_1(a, b)$—diverges if

$$t_n = O(|a|^n / \sqrt{n}), \tag{1.6}$$

or

$$(r_n) \text{ is bounded,}$$

or

$$(v_n) \text{ is bounded,}$$

since any one of such growth conditions implies eventual separation of the convergents. Our program for finally resolving the convergence domain of $\mathcal{R}_1(a, b)$ is now evident: We only need show that one of these growth conditions is true, for $a = be^{i\phi} \neq 0$ and $\cos^2\phi \neq 1$.

But first we offer in the next section a digression, an analysis of the (t_n) recurrence in the cases $\cos^2\phi = 1$. Note that [4] has established convergence/divergence theory for these cases $a = \pm b$; yet, the present analysis of sequence dynamics yields substantially more information on convergence rates.

2 The instance $a = \pm b$

Assume $a = \pm b$, so that $\mathcal{R}_1(a, a)$ is to be studied. For $a = b$ the recurrence (1.2) is blind to the parity of n (since $\kappa_n := a^2$, always), and the manipulations leading to the bound (1.5) are especially elegant and tractable. We note that divergence of $\mathcal{R}_1(a, a)$ for any pure-imaginary a (i.e. $a^2 \in (-\infty, 0)$) is established via a different approach in [4].

Let us assume $t_0 := 1$ so that fixing t_1 completely determines the entire sequence (t_n). An exponential generating function can be established as follows: Let $s_n := t_n/n!$

and consider

$$y(x) := \sum_{n=0}^{\infty} s_n x^n. \tag{2.1}$$

Using

$$\sum_{n=2}^{\infty} n^2 s_n x^n = \sum_{n=2}^{\infty} s_{n-1} x^n + a^2 \sum_{n=2}^{\infty} s_{n-2} x^n$$

we obtain

$$\sum_{n=2}^{\infty} n^2 s_n x^n = (x + a^2 x^2) \sum_{n=0}^{\infty} s_n x^n - t_0 x,$$

and hence that

$$\sum_{n=0}^{\infty} n^2 s_n x^n = (x + a^2 x^2) \sum_{n=0}^{\infty} s_n x^n + (t_1 - 1)x.$$

Thus

$$x \frac{d}{dx}(xy'(x)) = \sum_{n=0}^{\infty} n^2 s_n x^n = x(1 + a^2 x) y(x) + x(t_1 - 1).$$

Therefore $y := y(x)$ satisfies the differential equation

$$xy'' + y' - (1 + a^2 x)y = (t_1 - 1), \quad y(0) = 1, y'(0) = t_1.$$

This, for $t_1 = 1$ and general a with the help of *Maple*, has the solution

$$y(x) = e^{-ax} {}_1\mathrm{F}_1\left(\frac{a+1}{2a}; 1; 2ax\right) = e^{-ax} \sum_{n \geq 0} \left(\frac{a+1}{2a}\right)_n \frac{(2ax)^n}{n!^2},$$

where $(a)_n := a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol.

On equating coefficients we find that (here, by $t_n(t_0, t_1)$ is meant the recurrence solution starting with given t_0, t_1):

$$t_n(1, 1) = a^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-2)^k}{k!} \omega_k = a^n {}_2\mathrm{F}_1(-n, \omega; 1; 2), \tag{2.2}$$

where

$$\omega = \omega(a) := \frac{1 - 1/a}{2}$$

turns out to be an ubiquitous entity in our analysis.

Though (2.2) gives a finite form for t_n, it is not clear how to deduce other sequences (say, starting with $t_1 \neq 1$), moreover the large-n asymptotic behavior is nontrivial. One way to proceed on asymptotics is to use the rather intricate hypergeometric theory of J. Fields and Y. Luke [10, pp. 247–254], wherein the focus is upon expansions in the *parameters* rather than the argument. However, for our particular hypergeometric cases, a certain approach as outlined below is more specific and direct, less technical. Incidentally, and remarkably, by using the program ct, (described in [14, p. 112], and available on the home page of that book) one may actually *derive and prove* the recurrence (1.2) (with such assignments as $\kappa_n := -1$, all n) from the hypergeometric form (2.2), and likewise for (2.3) below.

Remarks. We have not been able to achieve similar success for $a \neq b$ by like methods. However

1. For general a, b the corresponding exponential generating function z can be neatly placed in the following coupled form:

$$t\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) - a^2 t y(t) = x(t), \quad t\frac{d^2}{dt^2}x(t) + \frac{d}{dt}x(t) - b^2 t x(t) = y(t),$$

   with $x(0) = 0$, $y(0) = 1$ and where y and x are the even and odd terms of z. Adding the odd and even terms when $a = b$ recovers the differential equation above. Note also that for $a = 0$ or $b = 0$ the underlying recursion is easy to solve explicitly.
2. Additionally, for $a = 1, b = i$, in terms of Bessel functions, we have a functional equation for the even part of the ordinary generating function:

$$y(t) = I_0(t) + \frac{\pi}{2}\int_0^t K(w,t)y(w)dw$$

   where the kernel is

$$K(w,t) := \int_w^t \{I_0(t)K_0(z) - K_0(t)I_0(z)\}\{J_0(z)Y_0(w) + Y_0(z)J_0(w)\}dz.$$

   The odd part has a similar equation. Note that K_0 and Y_0 have logarithmic singularities at 0 while $I_0(0) = J_0(0) = 1$.
3. Correspondingly, we can construct functional equations for general a and b by considering the appropriate differential equations

$$D_a(y) = x, D_b(x) = y.$$

   Let

$$D_a(y_a) = 0, y_a(0) = 1 \quad \text{and} \quad D_b(x_b) = 0, x_b(0) = 0,$$

   independently; then the corresponding Green's functions lead to the desired functional equation.

Inter alia, differential-equation techniques applied instead to the standard generating function can also yield similar results. One finds such oddities as the following, for the case $a = i$:

$$\sum_{n\geq 0} t_n(1,1)x^n = \frac{e^{\arctan\ x}}{\sqrt{1+x^2}} = \frac{1}{(1-ix)^{-\omega(i)^*}(1+ix)^{\omega(i)}}.$$

But still, the asymptotic behavior of the coefficients in such expansions is unclear. As we shall show, said coefficients can exhibit quite peculiar oscillations.

Armed with an exact form (2.2) for $t_n(1,1)$, we require an independent sequence in order to forge a general solution to recurrence (1.2). To this end, we found that a hypergeometric form related to that in (2.2) is

$$F_n(a) := a^n 2^{1-\omega} \frac{\Gamma(n+1)}{\Gamma(n+1+\omega)\Gamma(1-\omega)}\, {}_2\mathrm{F}_1\left(\omega, \omega; n+1+\omega; \frac{1}{2}\right). \tag{2.3}$$

which satisfies the recursion (1.2), for any parameter choice $a = b$ (for which, we recall, $\kappa_n := a^2$, always)—as can be checked in a computer algebra system. But here is an important observation: The same recurrence is also satisfied by $F_n(-a)$, since, after all, a^2 is the only a-dependent component of (1.2). This means that any solution of (1.2) is a superposition, such as

$$\begin{aligned} t_n(0,1) &= \alpha F_n(a) + \beta F_n(-a), \\ t_n(1,0) &= \gamma F_n(a) + \delta F_n(-a), \end{aligned} \tag{2.4}$$

where the constants $\alpha, \beta, \gamma, \delta$ can be written always in terms of the four constants $F_0(\pm a)$, $F_1(\pm a)$. We may also recast $t_n(1,1)$ as a superposition:

$$t_n(1,1) = \frac{1}{2}F_n(a) + \frac{1}{2}F_n(-a). \tag{2.5}$$

amounting to a hypergeometric identity involving the F_n and the right-hand side of (2.2). Such an identity, once known, can be derived from known transformation formulae [1, (2.3.12), (2.2.7) p. 68].

As to the asymptotic character of F_n for large n, denote

$$\Gamma_n(x) := \frac{n!n^x}{x(x+1)\cdots(x+n)} = \Gamma(x)\left\{1 + O\left(\frac{1}{n}\right)\right\},$$

and also

$$\omega := \omega(a), \quad \omega' := \omega(-a).$$

Then

$$F_n(a) = \frac{a^n 2^{\omega'}}{n^\omega} \frac{\Gamma_n(\omega)}{\Gamma(\omega)\Gamma(\omega')} \,{}_2\mathrm{F}_1\left(\omega, \omega; n+1+\omega; \frac{1}{2}\right).$$

Thus, the large-n behavior of (2.3) is

$$F_n(a) \sim \frac{2^{\omega'}}{\Gamma(\omega')} \frac{a^n}{n^\omega} \left\{1 + O\left(\frac{1}{n}\right)\right\}. \tag{2.6}$$

We now know from (2.4) the general asymptotic for arbitrary a, in the form

$$|t_n| = O\left(\frac{|a|^n}{n^{\mathrm{Re}(\omega)}} + \frac{|a|^n}{n^{\mathrm{Re}(\omega')}}\right) \tag{2.7}$$

a bound valid for *any* initial values t_0, t_1. This amounts to an analytic proof of [4, Theorem 5.1], that the Ramanujan fraction $\mathcal{R}_1(a, a)$ diverges for any pure-imaginary a; indeed, for such a we have $\mathrm{Re}(\omega) = \mathrm{Re}(\omega') = 1/2$, and the argument following (1.4) goes through.

We observe that the asymptotic relation (2.6) explains, finally, the interesting oscillation of a typical t_n sequence. For example, when $a = i$ we see that the t_n typically exhibit a fourfold quasi-oscillation, as (large) n runs through values modulo 4. (That is, when plotted versus n, the (real) sequence $t_n(1, 1)$ exhibits the "snaking" of four separate "necklaces.") In fact, for $a = i$ the detailed asymptotic is

$$\begin{aligned} t_n(1,1) &= \sqrt{\frac{2}{\pi}\cosh\frac{\pi}{2}}\frac{1}{\sqrt{n}}\left(1 + O\left(\frac{1}{n}\right)\right) \\ &\quad \times \begin{cases} (-1)^{n/2}\cos(\theta - \log(2n)/2) & \text{if } n \text{ is even} \\ (-1)^{(n+1)/2}\sin(\theta - \log(2n)/2) & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

where $\theta := \arg\Gamma((1+i)/2)$. This behavior is certainly difficult to infer directly from the recurrence (1.3), or even from our first hypergeometric form (2.2) for $t_1(1, 1)$.

For $\mathcal{S}(ri, ri)$ and any $r \neq 0$ we see a similar oscillatory behavior; indeed it is exactly in this case that (2.6) and (2.7) have precisely $\sqrt{n}$ growth.

3 The fraction convergents for $\mathcal{R}_1(a, a)$

The recurrence solutions (2.4) lead immediately to expressions for the convergents p_n/q_n to the $\mathcal{S}(a, a)$ fraction—and hence to the convergents, say P_n/Q_n of the original

$\mathcal{R}_1$ fraction, since $\mathcal{R}_1 =: 1/(1+\mathcal{S})$). In fact,

$$\frac{p_{n-1}}{q_{n-1}} = \frac{n!t_n(1,0)}{n!t_n(0,1)} = \frac{t_n(1,0)}{t_n(0,1)}$$
$$= \frac{F_1^- F_n^+ - F_1^+ F_n^-}{-F_0^- F_n^+ + F_0^+ F_n^-}, \tag{3.1}$$

where we denote $F_n^{\pm} := F_n(\pm a)$.

In an obvious sense we therefore have a closed form for the general convergent. Two interesting cases are $a = i$ and $a = 1$. In the former case the fraction diverges; in the latter case $\mathcal{R}_1 = \log 2$, as discussed in [3]. For the case $a = i$, our previous frustration in tracking the convergents numerically is now explained. In fact, on the basis of (3.1) and (2.6) the convergents to $\mathcal{S}(i, i)$ have asymptotic behavior, for n even,

$$\frac{p_{n-1}}{q_{n-1}} \sim -\frac{\cos\phi_0}{\cos\phi_1}\frac{\cos(\theta - \phi_1 - \log(2n)/2)}{\cos(\theta - \phi_0 - \log(2n)/2)},$$

where $\phi_{0,1}$ are certain (unequal) constant angles; while for n odd the second ratio of cosines is to be changed to a ratio of sines, of the same respective arguments. Also, in this asymptotic, it must be understood that $O(1/n)$ additive terms exist in both numerator and denominator.

More generally, (3.1) and (2.6) can be employed in this fashion to show that for nonzero real r, *the convergents p_n/q_n to the fraction $\mathcal{S}(ir, ir)$ are dense on the real axis*. It is possible, for example, to work out a (large) explicit integer n for which the n-th convergent to $\mathcal{S}(i, i)$ exceeds a googol (10^{100}). (Or for that matter, one can locate an n for which p_n/q_n is less than minus a googol.) Incidentally all this means that neither the even nor odd part of the fraction converges—a unique situation since for other, diverging cases of $|a| = |b|$ one still has separate, even/odd convergence [4].

The case $a = 1$, via relation (3.1) (and proper limit-taking, as $\omega' = \omega(-1) = 1$ and the ratio (3.1) needs be taken delicately) enjoys a striking, exact form for the general convergent of $\mathcal{R}_1$, namely

$$\frac{P_n}{Q_n} = \log 2 - \frac{(-1)^n}{2n+2}\,{}_2\mathrm{F}_1\left(1, 1; n+2, \frac{1}{2}\right).$$

We deduce that, remarkably, the right-hand side here is always rational. Moreover, we now know the precise convergence rate, namely $P_n/Q_n = \log 2 - (-1)^n/(2n + 2) + O(1/n^2)$, and such an observation may be a clue to acceleration algorithms for continued fractions of the Ramanujan type. Notice also how very much stronger this convergence knowledge is, over such as [3, Theorem 7.3].

For general a for which $\mathcal{R}_1(a, a)$ converges, let us with impunity force Re $(a) > 0$ (knowing that $\mathcal{R}_1(-a, -a) = \mathcal{R}_1(a, a)$), for which we have $\mathrm{Re}(\omega' := \omega(-a)) >$

$\mathrm{Re}(\omega(a))$ and therefore, from (3.1) and some manipulation,

$$\frac{P_n}{Q_n} = \frac{a}{1 + \frac{a}{1+\omega'} \frac{{}_2\mathrm{F}_1(\omega';\omega';2+\omega';1/2)}{{}_2\mathrm{F}_1(\omega';\omega';1+\omega';1/2)}} + O\left(\frac{1}{n^{\mathrm{Re}(1/a)}}\right) = \mathcal{R}_1(a,a) + O\left(\frac{1}{n^{\mathrm{Re}(1/a)}}\right), \tag{3.2}$$

so that the compound fraction involving the hypergeometric functions is the actual fraction value. It is interesting to compare said compound fraction with a previous exact evaluation from [3], namely

$$\mathcal{R}_1(a,a) = \frac{1}{\omega'}\,{}_2\mathrm{F}_1\,(\omega', 1; 1+\omega'; -1).$$

There is an interesting check of formula (3.2), namely we take $a = \infty$ so $\omega = \omega' = 1/2$ and

$$\frac{{}_2\mathrm{F}_1(1/2, 1/2; 3/2; 1/2)}{{}_2\mathrm{F}_1(1/2, 1/2; 5/2; 1/2)} = \frac{\pi/\sqrt{8}}{3/\sqrt{8}},$$

and sure enough, as explained in [3],

$$\mathcal{R}_1(\infty, \infty) = \frac{\pi}{2}.$$

It also follows from [3] that such hypergeometric ratios as appear in (3.2) can be put in closed form for any rational ω'.

4 Some initial matrix analysis

As precursor to what follows, let us do some elementary matrix analysis, focusing on the particular sequence (v_n) from (1.4). Consider the relevant two-step matrix iteration, with here, $\omega := \kappa_1/\kappa_0 = b^2/a^2$,

$$\begin{bmatrix} v_{2n} \\ v_{2n-1} \end{bmatrix} = Y_n \begin{bmatrix} v_{2n-2} \\ v_{2n-3} \end{bmatrix}, \tag{4.1}$$

where the Y matrix can be worked out to be

$$Y_n := \begin{bmatrix} \frac{4n^2+1/a^2}{4n^2-1/4} & \frac{\omega^n(2n-1)^2}{a(2n+1/2)((2n-1)^2-1/4)} \\ \frac{\omega^{-n}}{a(2n-1/2)} & \frac{(2n-1)^2}{(2n-1)^2-1/4} \end{bmatrix} \left(= I + \frac{1}{2an}\begin{bmatrix} 0 & \omega^n \\ \omega^{-n} & 0 \end{bmatrix} + O\left(\frac{1}{n^2}\right)\right).$$

But this means that

$$\begin{bmatrix} v_{2n} \\ v_{2n-1} \end{bmatrix} = Z_n \begin{bmatrix} v_0 \\ v_{-1} \end{bmatrix},$$

with

$$Z_n := \prod_{k=1}^{n} Y_k. \tag{4.2}$$

(Here and elsewhere, such a matrix product is interpreted as "left-handed," in that the $k = 1$ matrix is on the far right (see Remark to Theorem 6.1).)

Now

$$\det(Y_n)^{-1} = \left(1 - \frac{1}{(4n)^2}\right)\left(1 - \frac{1}{(4n-2)^2}\right)$$

is independent of (nonzero) (a, b) and so by a Wallis formula

$$\lim_{n\to\infty} \det(Z_n) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{(2n)^2}\right)^{-1} = \frac{\pi}{2}.$$

In spite of this easily derived determinantal limit—actually easier to see from the one-step iteration—we cannot yet assert that the matrices Z_n converge to a finite matrix, say Z_∞ Assume, though, that there *is* convergence, with

$$Z_\infty = \begin{bmatrix} A & C \\ B & D \end{bmatrix}, \qquad A, B, C, D \in \mathcal{C}. \tag{4.3}$$

Now let (u_n) denote the sequence (v_n) with $q_n \to p_n$ in (1.4). From the standard initial conditions for the $\mathcal{S}(a, b)$ fraction, we have

$$(u_{-1}, u_0, v_{-1}, v_0) = (1/\sqrt{\pi}, 0, 0, 2/(a\sqrt{\pi})), \tag{4.4}$$

implying that

$$\begin{aligned} v_{2n} &\sim 2A/(a\sqrt{\pi}), \quad & u_{2n} &\sim B/\sqrt{\pi}, \\ v_{2n-1} &\sim 2C/(a\sqrt{\pi}), \quad & u_{2n-1} &\sim D/\sqrt{\pi}. \end{aligned}$$

Such analysis leads immediately to

Theorem 4.1. *For nonzero complex parameters* (a, b) *with* $\omega := b^2/a^2$, *assume that the matrix* Z_n *converges to* Z_∞ *as in* (4.3). *Then the even/odd parts of* $\mathcal{S}(a, b)$ *both converge, respectively to*

$$\mathcal{S}^{(even)} = \frac{aB}{2A}, \quad \mathcal{S}^{(odd)} = \frac{aD}{2C},$$

which limits cannot be equal; in fact we have an explicit separation

$$\mathcal{S}^{(even)} - \mathcal{S}^{(odd)} = -\frac{a\pi}{4AC}.$$

The import of Theorem 4.1 is as follows: Whenever we can show that the infinite product Z_∞ exists, we have divergence of the $\mathcal{S}$ and perforce of $\mathcal{R}_1$, albeit with even/odd parts separately converging. (It was noted in [4] on the basis of established theory that the separate even/odd convergence holds for $a = be^{i\phi}$ with $\cos^2\phi \neq 1$, so any proof that Z_∞ exists repeats, at least, that result.)

5 Operator algebra theory

Our next foray into matrix analysis involves the matrix or sup-norm defined as follows. First, the norm of a complex number z will be the usual absolute value $|z|$, while the norm of a vector x of complex elements will be $|x| := (\sum |x_j|^2)^{1/2}$. For a complex $k \times k$ matrix M, we recall that the matrix or *operator norm* is defined as the "sup-norm" over all unit-norm vectors as

$$|M| := \sup_{|x|=1} |Mx|.$$

This norm is sometimes denoted $||M||_2$ [7], and has the requisite properties

$$\begin{aligned} |cM| &= |c||M|, \quad \text{for scalar } c, \\ |M+N| &\le |M| + |N|, \\ |MN| &\le |M||N|, \\ |M| &\le k \max|M_{ij}|. \end{aligned} \tag{5.1}$$

An important additional property is that

$$|M|^2 = \text{maximum eigenvalue of } M^\dagger M, \tag{5.2}$$

where † denotes adjoint (conjugate-transpose), and one may equally well use $MM^\dagger$.

We may apply sup-norm theory to establish divergence theorems for certain (a, b). Consider the (t_n) sequence, from (1.2). We define a vector τ_n,

$$\tau_n := \begin{bmatrix} t_{2n} \\ t_{2n-1} \end{bmatrix} = \frac{2n-2}{2n-1} E_n \begin{bmatrix} t_{2n-2} \\ t_{2n-3} \end{bmatrix}, \tag{5.3}$$

where the E_n matrix is

$$E_n := \begin{bmatrix} \dfrac{1 + b^2(2n-1)^2}{4n(n-1)} & \frac{a^2}{2n} \\ \dfrac{1}{2n-2} & a^2 \end{bmatrix}.$$

In turn, we have

$$E_n = F_n + O(1/n^2),$$

where

$$F_n = \begin{bmatrix} b^2 & \frac{a^2}{2n} \\ \frac{1}{2n} & a^2 \end{bmatrix}.$$

Being as, via the Wallis/Stirling formula,

$$\prod_{n=2}^{N} \frac{2n-2}{2n-1} \sim \sqrt{\frac{\pi}{4N}},$$

we have for a constant c

$$|\tau_N| \le \frac{c}{\sqrt{N}} \prod_{n=1}^{N} \left(|F_n| + O\left(\frac{1}{n^2}\right) \right). \tag{5.4}$$

However, from (5.2) we know $|F_n|$ is determined by the largest eigenvalue of $F_n F_n^\dagger$:

$$2|F_n|^2 = |a|^4 + |b|^4 + O\left(\frac{1}{n^2}\right) + \sqrt{\left(|b|^4 - |a|^4 + \frac{|a|^4 - 1}{(4n^2)}\right)^2 + \frac{|b^2 + |a|^4|^2}{n^2}}.$$

In this way the sup-norm theory applies to two significant cases. When $|a| \ne |b|$ we have

$$|F_n| = \max(|a|^2, |b|^2) + O\left(\frac{1}{n^2}\right),$$

leading via (5.4) and (1.5) to

Theorem 5.1. *If $|a| \ne |b|$ then any solution of recurrence* (1.2) *has*

$$|t_n| = O\left(\frac{\max(|a|, |b|)^n}{\sqrt{n}}\right),$$

and the convergents to $\mathcal{S}(a, b)$ have, for constant c,

$$\left|\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}}\right| > c \min\left(\left|\frac{a}{b}\right|, \left|\frac{b}{a}\right|\right)^{2n}.$$

Remark. It is already known [4] that for $|a| \ne |b|$ the $\mathcal{S}$ fraction converges on $\hat{\mathcal{C}}$. The sup-norm theory here bounds the convergence; this bound is consistent with the convergence rates for $|a| \ne |b|$ in [3, 4].

The second significant application of this sup-norm theory is the case $|a| = 1$, $b = i$, for which the eigenvalue calculation reduces to

$$|F_n| = 1 + O\left(\frac{1}{n^2}\right),$$

so that, from (5.4) and (1.5) we have

Theorem 5.2. *If* $|a| = 1, b = i$ *then any solution of recurrence* (1.2) *has*

$$|t_n| = O\left(\frac{1}{\sqrt{n}}\right),$$

and $\mathcal{S}(a, b)$*, perforce* $\mathcal{R}_1(a, b)$ *diverges.*

Remark. We have already taken care of the instance $(a, b) = (i, i)$ in our present Section 2, and in [4] we also handled $(a, b) = (i, i)$. Note that the present analysis also applies to sequences (t_n) for $b = i$ but with a chosen *randomly* from the unit circle.

A third application of sup-norm methods gives *a lower* bound on the accuracy of convergents. If we go back to the (v_n) sequence and observe that in (4.2), with $a \neq 0$, $|\omega| = 1$,

$$|Y_n| \leq \left| I + \frac{1}{2an}\Omega_n \right| + O\left(\frac{1}{n^2}\right).$$

where

$$\Omega_n := \begin{bmatrix} 0 & \omega^n \\ \omega^{-n} & 0 \end{bmatrix}, \tag{5.5}$$

then a similar eigenvalue analysis on $(I + \Omega_n/(2an))^\dagger (I + \Omega_n/(2an))$ yields, for constants c_1, c_2,

$$|Z_N| \leq c_1 \sqrt{\prod_{n=1}^{N} \left(1 + \frac{1}{n}\left|\mathrm{Re}\left(\frac{1}{a}\right)\right|\right)} < c_2 N^{|\mathrm{Re}(1/a)|/2}.$$

Therefore $|v_n| < c_2 n^{|\mathrm{Re}(1/a)|/2}$, so by the last relation of (1.5) we obtain

Theorem 5.3. *If* $|a| = |b|$ *then the convergents to* $\mathcal{S}(a, b)$ *have, for positive constant* c

$$\left|\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}}\right| \geq \frac{c}{n^{|\mathrm{Re}(1/a)|}}.$$

Remark. This result is consistent with our exact, hypergeometric analysis of Section 2 when $a = b$. It is also a corollary of Theorem 5.3 that—as we have seen several times already—fractions $\mathcal{S}$, $\mathcal{R}_1$ diverge for $a^2 = b^2 \in (-\infty, 0)$. Indeed, for such cases the exponent of n in the bounding right-hand side is zero, so the fraction cannot converge. In spite of the allure of this sup-norm theory, it appears difficult to generalize Theorem 5.2 to more general cases $|a| = |b|$ using sup-norm methods alone. Evidently, deeper results are required to resolve finally the convergence problem for $\mathcal{R}_1$. Accordingly, with a view to Theorem 4.1, our next foray will address not just sup-norm bounds on, but actual convergence of matrix products.

6 Some deeper matrix analysis

We know now that S and therefore $\mathcal{R}_1$ diverges if the product matrix Z_n in (4.2) converges to a finite complex matrix. To this end, we shall now prove a general "perturbation" theorem which will turn out to have application even beyond our present analysis.

Theorem 6.1. *Let (a_n), (b_n) be sequences of $k \times k$ complex matrices. Suppose that $\prod_{j=1}^{n} a_j$ converges to L as $n \to \infty$ where L is invertible, and that $\sum_{j=1}^{\infty} |b_j| < \infty$. Then*

$$\prod_{j=1}^{n} (a_j + b_j)$$

converges to a finite complex matrix as $n \to \infty$.

Remark. Again, matrix products involving symbology $\prod_{k \geq 1}$ are interpreted as having the $(k = 1)$ matrix at the far right. However, oppositely defined products can easily be handled via transposition of whole products.

Note that in one dimension if the limit is singular (i.e., zero) one talks about divergence to zero.

We shall require a series of lemmas, whose development could be stream-lined by applying more functional analytic ideas—for example, Lemma 6.2 is a consequence of the Banach open mapping theorem along with the principle of uniform boundedness, or of the stability of the condition number—but we opt to leave everything explicit.

Lemma 6.2. *Let (l_n) be a sequence in the set of $k \times k$ complex matrices $M_{k\times k}$. Suppose that $(l_n) \to L$, where L is an invertible matrix. Then there exists an $N \in Z^+$ and a $K \in R^+$ such that for all $n \in Z^+$, $n \geq N$ implies*

$$|bl_n| \geq K|b| \text{ for all } b \in M_{k\times k}.$$

Proof: Since the invertible elements form an open set in $M_{k\times k}$, there is an $\varepsilon > 0$ such that the set $B_\varepsilon = \{x \in M_{k\times k} : |x - L| \leq \varepsilon\}$ consists entirely of invertible elements.

Let $U = \{x \in M_{k\times k} : |x| = 1\}$. Define $\Phi : U \times B_\varepsilon \to R$ by

$$\Phi(u, x) = |ux| \text{ for all } (u, x) \in U \times B_\varepsilon.$$

Then Φ is a continuous function on the compact set $U \times B_\varepsilon$, so it has a minimum value K. Since x is invertible, and $u \neq 0$ implies $|ux| > 0$, we see that $K > 0$. We have

$$\left|\frac{b}{|b|} \cdot x\right| \geq K \text{ for all } b \in (M_{k\times k}\backslash\{0\}), x \in B_\varepsilon,$$

and hence

$$|bx| \geq K|b| \text{ for all } b \in M_{k\times k}, x \in B_\varepsilon.$$

Since $(l_n) \to L$, there is an $N \in Z^+$ such that $l_n \in B_\varepsilon$ for $n \geq N$. Then $n \geq N$ implies $|bl_n| \geq K|b|$ for all $b \in M_{k\times k}$. □

Let us say that a matrix product is *tail-Cauchy* if $\prod_{j=p+1}^{q} a_j \to I$ as $q > p \to \infty$ (i.e., both tend to infinity). We note for future use that every tail-Cauchy sequence is bounded.

Lemma 6.3. *Let (a_n) be a sequence in $M_{k\times k}$.*

a. Every tail-Cauchy product converges;
b. Conversely, suppose $(\prod_{j=1}^{n} a_j) \to L$, where L is invertible, then (a_n) is tail-Cauchy.

Proof:

(a) As observed every tail-Cauchy product is bounded. Indeed, there is an $M \in Z^+$ such that

$$q > M \Longrightarrow \left|\prod_{j=M+1}^{q} a_j - I\right| < 1 \Longrightarrow \left|\prod_{j=1}^{q} a_j\right| < 2\left|\prod_{j=1}^{M} a_j\right|.$$

Let $R := 2|\prod_{j=1}^{M} a_j|$ (or any upper bound), and let $\varepsilon \in R^+$. By the tail-Cauchy property, we can choose $P > M$ such that

$$q > p \geq P \Longrightarrow \left|\prod_{j=p+1}^{q} a_j - I\right| < \frac{\varepsilon}{R}.$$

Then

$$q > p \geq P \Longrightarrow \left|\prod_{j=1}^{q} a_j - \prod_{j=1}^{p} a_j\right| \leq \left|\prod_{j=p+1}^{q} a_j - I\right| \left|\prod_{j=1}^{p} a_j\right| < \frac{\varepsilon}{R} \cdot R = \varepsilon.$$

Thus $(\prod_{j=1}^{n} a_j)$ is a Cauchy sequence, and hence it converges.

(b) Conversely, suppose the product converges to an invertible limit. By Lemma 6.2 there is an $N \in Z^+$ and $K \in R^+$ such that

$$p \geq N \Longrightarrow \left| x \prod_{j=1}^{p} a_j \right| \geq K|z| \text{ for all } x \in M_{k\times k}.$$

Since $(\prod_{j=1}^{n} a_j)$ converges, it is Cauchy and so there is an $M \in Z^+$ such that

$$q > p \geq M \Longrightarrow \left| \prod_{j=1}^{q} a_j - \prod_{j=1}^{p} a_j \right| < \varepsilon K \Longrightarrow \left| \left(\prod_{j=p+1}^{q} a_j - I \right) \prod_{j=1}^{p} a_j \right| < \varepsilon K.$$

Let $P = \max\{M, N\}$. Then, as required

$$q > p \geq P \Longrightarrow \left| \prod_{j=p+1}^{q} a_j - I \right| \leq \left| \left(\prod_{j=p+1}^{q} a_j - I \right) \prod_{j=1}^{p} a_j \right| \Big/ K < \varepsilon.$$

□

Lemma 6.4 (Key inequality). *Let (a_n), (b_n) be sequences of $k \times k$ complex matrices. Suppose p in Z^+ is given such that*

$$q > r \geq p \longrightarrow \left| \prod_{j=r+1}^{q} a_j \right| \leq M < \infty. \tag{6.1}$$

Then for positive n,

$$\left| \prod_{j=p+1}^{p+n} (a_j + b_j) - \prod_{j=p+1}^{p+n} a_j \right| \leq M \left(\prod_{j=p+1}^{p+n} (1 + M|b_j|) - 1 \right).$$

Proof: For $n \in Z^+$, let $Z_n = \{1, 2, \ldots, n\}$, and for $p, n \in Z^+$ and $S \subset Z_n$ let

$$\pi_S^{p,n} := x_n x_{n-1} \ldots x_2 x_1$$

where

$$x_j := b_{p+j} \text{ if } j \in S \text{ but } x_j := a_{p+j} \text{ if } j \notin S.$$

Then

$$\prod_{j=p+1}^{p+n} (a_j + b_j) - \prod_{j=p+1}^{p+n} a_j = \sum_{S \subset Z_n, S \neq \emptyset} \pi_S^{p,n}.$$

Let $|S|$ denote the number of elements in S. By hypothesis (6.1) we have $M > 1$ such that

$$q > r \geq p \Longrightarrow \left| \prod_{j=r+1}^{q} a_j \right| \leq M.$$

If $n \in Z^+$ and $S \subset Z_n$, then since there are at most $|S| + 1$ 'runs' of a_{p+j}'s

$$\left| \pi_S^{p,n} \right| \leq M^{|S|+1} \prod_{j \in S} |b_{p+j}|,$$

Hence,

$$\begin{aligned}
\left| \prod_{j=p+1}^{p+n} (a_j + b_j) - \prod_{j=p+1}^{p+n} a_j \right| &\leq \sum_{k=1}^{n} \sum_{S \subset Z_n, |S|=k} \left| \pi_S^{p,n} \right| \leq \sum_{k=1}^{n} \sum_{S \subset Z_n, |S|=k} M^{k+1} \prod_{j \in S} |b_{p+j}| \\
&= M \left(\sum_{k=1}^{n} \sum_{S \subset Z_n, |S|=k} \prod_{j \in S} (M|b_{j+p}|) \right) \\
&= M \sum_{S \subset Z_n, S \neq \emptyset} \prod_{j \in S} (M|b_{j+p}|) \\
&= M \left(\prod_{k=1}^{n} (1 + M|b_{k+p}|) - 1 \right) \\
&= M \left(\prod_{j=p+1}^{p+n} (1 + M|b_j|) - 1 \right).
\end{aligned}$$

□

We may now establish the perturbation result

Proof of Theorem 6.1: By Lemma 6.3(b), (a_n) has a tail-Cauchy product and so hypothesis (6.1) holds for large p. Also, since $\sum |b_j|$ is convergent, $\prod (1 + M|b_j|)$ converges. Hence Lemma 6.4, shows that.

$$\left| \prod_{j=p+1}^{q} (a_j + b_j) - \prod_{j=p+1}^{q} a_j \right| \to 0$$

when $q > p \to \infty$. By the triangle inequality, $(a_n + b_n)$ has a tail-Cauchy product. An appeal to Lemma 6.3(a) concludes the proof of Theorem 6.1. □

The importance of the perturbation Theorem 6.1 is evident: When analyzing the dynamics of recurrence sequences, we may—with impunity—discard certain troublesome terms in the overall matrix analysis. An immediate application of this idea appears in the next section.

Examples. It is instructive to note counterexamples to broader variants of Theorem 6.1, or Lemma 6.3(b), for example the invertibility of L is mandatory. For $n \in Z^+$, define

$$a_n := \begin{bmatrix} 1 & n^3 \\ 0 & \frac{1}{(n+1)^3} \end{bmatrix} \qquad b_n := \begin{bmatrix} 0 & 0 \\ \frac{1}{n^2} & 0 \end{bmatrix}.$$

Then we see by induction that

$$\prod_{j=1}^{n} a_j = \begin{bmatrix} 1 & \sum_{j=0}^{n-1} \frac{1}{(j!)^3} \\ 0 & \frac{1}{((n+1)!)^3} \end{bmatrix},$$

so $(\prod_{j=1}^{n} a_j)$ converges to a singular matrix, although the finite products are all nonsingular. However $\prod_{j=1}^{n}(a_j + b_j)$ *diverges* since $\det \prod_{j=1}^{n}(a_j + b_j) = \prod_{j=1}^{n}(\frac{1}{(j+1)^3} - j)$ which clearly diverges. Also (a_n) is clearly divergent.

Another example (from the scalar ($k = 1$) domain) is $a_n := 2^n$, for n even and $a_n := 4^{-n}$ for n odd, yield convergent (to zero) products that are not tail-Cauchy and such that adding $b_n := 2^{-n}$ provides a summable perturbation for which $\prod_{n>0}(a_n + b_n)$ oscillate unboundedly. Indeed, for $P_n := \prod_{j=1}^{n}(a_j + b_j)$ we have $P_{2n} \to \infty$ while $P_{2n+1} \to 0$.

Note also that $a_n := 1$ and $b_n := -x^2/n^2$ provides a summable perturbation for which $\prod_{n>0}(a_n + b_n) = \sin(\pi x)/(\pi x)$ has zeros.

We conclude the section by recording the operator theoretic analogue of Theorem 6.1.

Corollary 6.5. *Theorem* 6.1 *holds in any Banach algebra with unit, and so in particular, for the bounded linear operators endowed with the uniform norm.*

Proof: We replace Lemma 6.2 by the standard fact [12, p. 310] that the mapping $x \mapsto x^{-1}$ is continuous at any invertible element. $\square$

7 Exponential-sum analysis

In view of the perturbation Theorem 6.1, when $|\omega| = 1$ we may write our key matrix product (4.2) relevant to the (v_n) sequence as

$$Z_N = \prod_{n=1}^{N} \left(I + \frac{1}{2an} \Omega_n + O\left(\frac{1}{n^2}\right)\right),$$

where

$$\Omega_n := \begin{bmatrix} 0 & \omega^n \\ \omega^{-n} & 0 \end{bmatrix}.$$

In this way, Theorem 6.1 essentially allows simplification of the Y_n matrix, because $Y_n \sim I + \Omega_n/(2an)$. As soon as we can show

$$U_N := \prod_{n=1}^{N} \left(I + \frac{1}{2an} \Omega_n \right)$$

converges for $a \neq 0, \omega := b^2/a^2 \neq 1, |\omega| = 1$, then the analysis has been brought full-circle, with $\mathcal{R}_1(a, b)$ diverging for such parameters via Theorem 4.1, so that the convergence region for $\mathcal{R}_1$ is finally resolved. (Note that, in view of the perturbation Theorem 6.1, if some U_N is singular we may take the product from some sufficiently large $n = n_0$ to ensure convergence to an invertible U_∞.)

With a view to calculation of the U_N products, we introduce certain exponential sums—which might be called *multiple-Lerch sums*—via

$$T_n(N, \omega) := \sum_{N \geq j_n > j_{n-1} > \cdots > j_1 \geq 1} \frac{\omega^{j_n - j_{n-1} + \cdots}}{j_n j_{n-1} \cdots j_1}, \tag{7.1}$$

where the indices are alternating in the exponent of ω, and the sum is considered empty (0) if $n > N$, with also the assignment $T_0(N, \omega) := 1$. These sums are generally very difficult to cast into closed form, even for the full sum $T_n(\infty, \omega)$; however, we shall be able to establish useful upper bounds—indeed, we shall see that in our preferred scenario $|\omega| = 1, \omega \neq 1$, the sum $T_n(\infty, \omega)$ decays very rapidly in n.

Evidently,

$$U_N = \begin{bmatrix} \alpha_N(\omega) & \beta_N(\omega) \\ \beta_N(\omega^{-1}) & \alpha_N(\omega^{-1}) \end{bmatrix},$$

where

$$\alpha_N(\omega) := \sum_{n=0}^{\infty} (2a)^{-2n} T_{2n}(N, \omega),$$

$$\beta_N(\omega) := \sum_{n=0}^{\infty} (2a)^{-(2n+1)} T_{2n+1}(N, \omega). \tag{7.2}$$

Note that these sums are manifestly finite, since $T_m(N, \omega)$ vanishes for $m > N$. Our goal is to show that both α_N, β_N converge to finite complex values as $N \to \infty$, regardless of complex $a \neq 0$ with $|\omega| = 1, \omega \neq 1$.

Examples . More explicitly let

$$L_\psi(\omega, n) := T_n(\infty, \omega) = \sum_{j_n > j_{n-1} > \cdots > j_1 \geq 1} \frac{\omega^{j_n - j_{n-1} + \cdots}}{j_n j_{n-1} \cdots j_1}.$$

Observe that $L_\psi(\omega,1) = -\log(1-\omega)$ for any $\omega \neq 1$, $|\omega| \leq 1$. For the case $\omega = -1$, $L_\psi(-1,n)$ is evaluated in [2] and elsewhere via $L_\psi(-1,n) = \zeta(-1,-1,\ldots,-1)$ (repeated n times) which is the coefficient of t^n in

$$A(t) := \frac{\sqrt{\pi}}{\Gamma\left(1+\frac{t}{2}\right)\Gamma\left(\frac{1}{2}-\frac{t}{2}\right)}.$$

Our subsequent analysis will reveal that said coefficient decays very rapidly as $n \to \infty$.

Analyzing more general L_ψ seems substantially more difficult. For example, taking $\omega := i$ we obtain

$$L_\psi(i,1) = i\int_0^1 \frac{dx}{1-ix} = -\log(1-i) = \frac{\pi}{4} + \frac{i}{2}\ln(2) = i\,\mathrm{Li}_1\left(\frac{1}{2}-\frac{i}{2}\right),$$
$$L_\psi(i,2) = \sigma(1) = \frac{5\pi^2}{96} - \frac{1}{8}\ln^2(2) + i\left(\frac{\pi}{8}\ln(2) - \mathrm{G}\right) = -\mathrm{Li}_2\left(\frac{1}{2}-\frac{i}{2}\right),$$

where $G := \sum_{n\geq 0}(-1)^n/(2n+1)^{-2}$ is the Catalan constant, and Li_n is the standard polylogarithm of order n. This last relation follows from

$$\begin{aligned}\sigma(t) &:= \int_0^t\int_0^x \frac{1}{(y+i)(x+i)x}dy\,dx\\ &= \frac{\zeta(2)}{4} + \frac{1}{2}\mathrm{Li}_2\left(\frac{2t}{t+i}\right) - \frac{1}{2}\mathrm{Li}_2\left(\frac{1+it}{2}\right) - \frac{1}{4}\mathrm{Li}_2(-t^2)\\ &\quad + \frac{1}{4}\ln(2)\ln\left(\frac{t^2+1}{2}\right) - \frac{3}{16}\ln^2(t^2+1)\\ &\quad + \arctan(t)\left\{\frac{i}{4}\ln(4t^2+4) - \frac{1}{4}\arctan(t)\right\}.\end{aligned}$$

With significant symbolic computational help we can derive

$$\begin{aligned}L_\psi(i,3) = &-\frac{1}{4}\int_0^1 \frac{\ln^2\left(\frac{t^2+1}{2}\right)}{1-t}dt + \frac{1}{16}\ln^3(2) - \frac{7}{32}\zeta(3) - \frac{5\pi^2}{192}\ln(2)\\ &- i\left\{\frac{7\pi^3}{128} - \frac{\mathrm{G}}{2}\ln(2) + \frac{\pi}{32}\ln^2(2) + 2\,\mathrm{Im}\,\mathrm{Li}_3\left(\frac{1-i}{2}\right)\right\}.\end{aligned}$$

The real part is also expressible as:

$$\frac{5\pi^2}{192}\ln(2) - \frac{1}{4}\zeta(3) - \frac{1}{48}\ln^3(2) - \frac{1}{4}\sum_{n=1}^{\infty}\frac{2^n\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)}{n^2\binom{2n}{n}},$$

and one may use

$$\int_0^1 \frac{\ln^2\left(\frac{t^2+1}{2}\right)}{1-t}dt = \frac{1}{2}\ln^3(2) - \frac{5\pi^2}{24}\ln(2) + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\sum_{j=1}^{n-1}\frac{1}{j}\sum_{k=1}^{2n}\frac{1}{k}.$$

Consider now a general, truncated, constrained exponential sum

$$\begin{aligned} W_n(N;\rho,\dots,\rho_n) &:= \sum_{N\ge j_n>j_{n-1}>\cdots j_1\ge 1}\frac{\rho_n^{j_n}\dots\rho_1^{j_1}}{j_n j_{n-1}\dots j_1} \\ &= \left(\prod_{k=1}^{n}\rho_k\right)\int_0^1\cdots\int_0^1 dx_1\dots dx_n\, S_n(N;\rho_1x_1,\dots,\rho_nx_n), \end{aligned} \tag{7.3}$$

(when the integral exists), where we define

$$S_n(N;z_1,\dots,z_n) := \sum_{N>k_n>\cdots k_1\ge 0} z_n^{k_n} z_{n-1}^{k_{n-1}}\cdots z_1^{k_1}.$$

Note that the full constrained exponential sum is interpreted as $W_n(\infty,\rho_1,\dots,\rho_n)$ which is the integral (7.3) with $S_n(\infty;\rho_1x_1,\dots,\rho_nx_n)$ used on the right side. Note now, the combinatorial matter,

$$S_n(\infty;z_1,\dots,z_n) = \frac{z_n^{n-1}}{1-z_n}\frac{z_{n-1}^{n-2}}{1-z_nz_{n-1}}\cdots\frac{1}{1-z_nz_{n-1}\dots z_1}. \tag{7.4}$$

Our exponential sum (7.1) now takes the form

$$T_n(N,\omega) = W_n(N;\omega^{\pm 1},\dots,\omega^{-1},\omega),$$

so that the full multiple-Lerch sum is

$$\begin{aligned} T_n(\infty,\omega) &= \sum_{j_n>j_{n-1}>\cdots>j_i\ge 1}\frac{\omega^{j_n-j_{n-1}+\cdots}}{j_n j_{n-1}\dots j_1} \\ &= \omega^{n \bmod 2}\int_0^1\cdots\int_0^1 dx_1\dots dx_n\, S_n(\infty;\omega^{2(n \bmod 2)-1}x_1,\dots,\omega^{-1}x_{n-1},\omega x_n) \\ &= \omega^{\lfloor (n+1)/2\rfloor}\int_0^1\cdots\int_0^1 \frac{x_n^{n-1}}{1-\omega x_n}\frac{x_{n-1}^{n-2}}{1-x_nx_{n-1}}\frac{x_{n-2}^{n-3}}{1-\omega x_nx_{n-1}x_{n-2}} \\ &\quad \cdots dx_1\dots dx_n. \end{aligned} \tag{7.5}$$

We now establish four lemmas, with a view to bounding both $T_n(\infty,\omega)$ and $T_n(\infty,\omega) - T_n(N,\omega)$ for finite N:

Lemma 7.1. *The truncation error term*

$$E_n(N; z_1, \dots, z_n) := S_n(\infty, z_1 \dots, z_n) - S_n(N, z_1, \dots, z_n)$$

is given by ($S_0 := 1$, and assuming all $|z_k| < 1$)

$$E_n(N) = \sum_{j=1}^{n} (-1)^{j-1} \frac{S_{n-j}(\infty, z_1, \dots, z_{n-j})(z_n \dots z_{n-j+1})^N}{(1 - z_{n-j+1})(1 - z_{n-j+1} z_{n-j+2}) \cdots (1 - z_{n-j+1} \dots z_n)}.$$

Proof: Suppose that $|z_k| < 1$ for all $k \geq 1$. Let $S_n := S_n(\infty, z_1, \dots, z_n)$, and

$$c_{k,n} := \frac{(-1)^{k+1} S_{k-1}}{\prod_{j=k}^{n} (1 - z_k z_{k+1} \dots z_j)}.$$

The required conclusion is then

$$E_n(N) = \sum_{k=1}^{n} c_{k,n} \left(\prod_{j=k}^{n} z_j \right)^N . \tag{7.6}$$

Recursively, for $n > 1$, we have

$$E_n(N) = \frac{z_n^N}{1 - z_n} S_{n-1} - z_n^N \sum_{m=0}^{\infty} z_n^m E_{n-1}(m + N)$$

and

$$E_1(N) = \frac{z_1^N}{1 - z_1}.$$

We now prove (7.6) by induction. Direct calculation shows that (7.6) holds for $n = 2$ and $n = 3$. Suppose it holds with $n - 1$ in place of n, so that

$$E_{n-1}(N) = \sum_{k=1}^{n-1} c_{k,n-1} \left(\prod_{j=k}^{n-1} z_j \right)^N ,$$

and hence

$$z_n^N \sum_{m=0}^{\infty} z_n^m E_{n-1}(m + N) = z_n^N \sum_{k=1}^{n-1} c_{k,n-1} \sum_{m=0}^{\infty} z_n^m \left(\prod_{j=k}^{n-1} z_j \right)^{N+m}$$

$$= \sum_{k=1}^{n-1} c_{k,n-1} \frac{\left(\prod_{j=k}^{n} z_j \right)^N}{1 - \prod_{j=k}^{n} z_j} = - \sum_{k=1}^{n-1} c_{k,n} \left(\prod_{j=k}^{n} z_j \right)^N .$$

Thus, by the recursion,

$$E_n(N) = \frac{z_n^N}{1 - z_n} S_{n-1} + \sum_{k=1}^{n-1} c_{k,n} \left(\prod_{j=k}^{n} z_j \right)^N = \sum_{k=1}^{n} c_{k,n} \left(\prod_{j=k}^{n} z_j \right)^N ,$$

as desired. □

We shall eventually use the idea that the truncation error in the multiple-Lerch sum itself has a representation

$$T_n(\infty, \omega) - T_n(N, \omega) = \omega^{n \bmod 2} \int_0^1 \cdots \int_0^1 dx_1 \ldots dx_n \, E_n(N; \omega^{2(n \bmod 2)-1} x_1, \ldots, \omega^{-1} x_{n-1}, \omega x_n). \tag{7.7}$$

This integral can be given in a more explicit form such as (7.5), using Lemma 7.1 and the symbolic form (7.4).

Now we define for ω on the unit complex circle,

$$\sigma(\omega) := \sin(\min(\pi/2, |\arg(\omega)|)).$$

Lemma 7.2. *For positive integer p, with $q = p$ or $p - 1$, and $|\omega| = 1$, real $\lambda \in [0, 1]$ define*

$$I(p, q, \omega, \lambda) := \int_0^1 \int_0^1 \frac{x^p y^q}{(1 - \omega\lambda x)(1 - \lambda x y)} dx dy.$$

Then

$$|I| \le \frac{1}{\sigma(\omega) p}.$$

Proof: We have

$$|I| \le \sup_{0 \le \rho \le 1} \frac{1}{|1 - \omega\rho|} \sum_{m=1}^{\infty} \frac{1}{p + m} \frac{1}{q + m}.$$

Note first that $|1 - \omega\rho|^2 = 1 - 2\rho \cos \arg(\omega) + \rho^2 \ge \sigma(\omega)^2$. The final sum is easily estimated by

$$\sum_{m=1}^{\infty} \frac{1}{(p + m)^2} < \sum_{m=1}^{\infty} \frac{1}{(p + m)(p - 1 + m)} = \frac{1}{p}.$$

□

We also have

Lemma 7.3. *For* $|\omega| = 1$*, the truncated multiple-Lerch sum has*

$$|T_n(N, \omega)| < \frac{2^N}{n!}.$$

Proof: Clearly, by considering the possible indices j_m in (7.1),

$$|T_n(N, \omega)| \le \frac{\binom{N}{n}}{n!}$$

and the result follows. □

These observations lead to sufficiently strong exponential-sum bounds:

Lemma 7.4. *For* $|\omega| = 1, \omega \neq 1$ *the multiple-Lerch sum* (7.1) *has*

$$|T_n(\infty, \omega)| \le \frac{1}{\sigma^{\lfloor (n+1)/2 \rfloor}} \frac{1}{(n-1)!!},$$

and for $n < N$*,*

$$|T_n(\infty, \omega) - T_n(N, \omega)| \le \frac{1}{\sigma^{n/2+1}} \frac{1}{(n-1)!!} \frac{2n}{N-n},$$

where $\sigma := \sigma(\omega)$ *and the double-factorial is defined by* $q!! := q(q-2)\cdots((q \bmod 2)+1)$ *for positive integers* q*, with* $0!! = (-1)!! := 1$.

Proof: We think of the integral representation (7.5) as a chain of $\lfloor n/2 \rfloor$ double integrals of the I-type of Lemma 7.2, with possibly one integral (over x_1) left over. Lemma 7.2 thus gives immediately

$$|T_n(\infty, \omega)| \le 1/\sigma \cdot 1/(\sigma \cdot (n-1) \cdot \sigma \cdot (n-3) \cdots),$$

which is the desired bound for $N = \infty$. The integral (7.7) is more intricate, but happily the result of Lemma 7.1 is that E_n becomes separated with respect to integration variables $x_1, \ldots, x_{n-j}$ (via the S_{n-j} terms) and the variables $x_{n-j+1}, \ldots, x_n$. So the integral (7.7) separates to give, again via Lemma 7.2,

$$\begin{aligned}
|T_n(\infty, \omega) - T_n(N, \omega)| &\le \sum_{j=1}^{n} \frac{1}{\sigma^{\lfloor (n-j+1)/2 \rfloor}} \frac{1}{(n-j-1)!!} \frac{1}{\sigma^{\lfloor (j+1)/2 \rfloor}} \frac{1}{N^{\lfloor (j+1)/2 \rfloor}} \\
&\le \frac{1}{\sigma^{n/2+1}} \left(\frac{1}{N} \left(\frac{1}{(n-2)!!} + \frac{1}{(n-3)!!} \right) \right. \\
&\quad \left. + \frac{1}{N^2} \left(\frac{1}{(n-4)!!} + \frac{1}{(n-5)!!} \right) + \cdots \right)
\end{aligned}$$

$$\leq \frac{1}{\sigma^{n/2+1}} \frac{1}{(n-1)!!} \left(\frac{2n}{N} + \frac{2n^2}{N^2} + \cdots \right)$$
$$\leq \frac{1}{\sigma^{n/2+1}} \frac{1}{(n-1)!!} \frac{2n}{N} \frac{1}{1-n/N},$$

which proves the second bound of the theorem. □

Next we define functions (recall $0!! = (-1)!! := 1$)

$$F(z) := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n-1)!!} = 1 + ze^{z^2/2} \int_0^z e^{-t^2/2} dt,$$
$$G(z) := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!!} = e^{z^2/2}.$$

The sums α_N, β_N in (7.2) may now be addressed. We shall say that a matrix A *dominates* B if $|A_{jk}| \geq |B_{jk}|$ for all index pairs j, k.

Theorem 7.5. *For $a \neq 0$, $|\omega| = 1$, $\omega \neq 1$ the matrix*

$$U_N := \prod_{n=1}^{N} \left(I + \frac{1}{2an} \Omega_n \right),$$

where

$$\Omega_n := \begin{bmatrix} 0 & \omega^n \\ \omega^{-n} & 0 \end{bmatrix},$$

converges as $N \to \infty$. Moreover, U_∞ is dominated by the matrix (with $\sigma := \sigma(\omega)$, $\rho := 1/|2a\sqrt{\sigma}|$):

$$\begin{bmatrix} F(\rho) & \rho G(\rho)/\sqrt{\sigma} \\ \rho G(\rho)/\sqrt{\sigma} & F(\rho) \end{bmatrix}.$$

Proof: Consider for the moment the α_N element from (7.2), assume integer $M \in [1, N/2)$, and decompose (recalling that α_N, β_N are actually finite sums, as $T_m(N, \omega) := 0$ for $m > N$):

$$\alpha_N(\omega) = \sum_{n=0}^{\lfloor N/2 \rfloor} \frac{1}{(2a)^{2n}} T_{2n}(N, \omega)$$

$$= \left(\sum_{0 \le n \le M} + \sum_{n=M+1}^{\lfloor N/2 \rfloor} \right) \frac{1}{(2a)^{2n}} T_{2n}(N, \omega)$$

$$= \sum_{0 \le n \le M} \frac{1}{(2a)^{2n}} T_{2n}(\infty, \omega) + \sum_{0 \le n \le M} \frac{1}{(2a)^{2n}} (T_{2n}(N, \omega) - T_{2n}(\infty, \omega))$$

$$+ O\left(\sum_{n=M+1}^{\lfloor N/2 \rfloor} \frac{1}{(2a)^{2n}} \frac{2^N}{(2n)!} \right).$$

It is evident from the first bound of Lemma 7.4 that the first sum here converges absolutely as $M \to \infty$. The second sum, by the second bound of Lemma 7.4, is bounded by the same F factor times $4\sqrt{\sigma} M/(N - 2M)$, and the big-$O$ term follows from Lemma 7.3. Now, for positive real x and the choice $M = \lceil \frac{1}{2} \frac{N}{\log N} - 1 \rceil$ with also $2M^2 > x^2$,

$$2^N \sum_{n > M} \frac{x^{2n}}{(2n)!} < 2^N \frac{x^{2M+2}}{(2M+2)!} \frac{1}{1 - x^2/(4M^2)} < 2 \left(\frac{3x \log N}{N^{1-\log 2}} \right)^{N/\log N},$$

where the last inequality arises from the Sterling bound $k! > (k/3)^k$. For our current choice of M we thus have

$$\alpha_N(\omega) = \sum_{0 \le n \le M} \frac{1}{(2a)^{2n}} T_{2n}(\infty, \omega) + O\left(\frac{1}{\log N} \right).$$

This means that the $\alpha_N(\omega)$ matrix element converges to the value

$$\alpha_\infty(\omega) := \sum_{n \ge 0} \frac{1}{(2a)^{2n}} T_{2n}(\infty, \omega),$$

which is in turn bounded in magnitude by $F(\rho)$. The same analysis applies to $\alpha_N(\omega^{-1})$ (note $\sigma(\omega) = \sigma(\omega^{-1})$). The same overall procedure works for β_N sums together with use of the G function, and the theorem is in this way proved. □

Corollary 7.6. *For $a \ne 0$, $\omega := b^2/a^2 \ne 1$, $|\omega| = 1$ the Ramanujan fraction $\mathcal{R}_1(a, b)$ diverges. In particular, the even/odd parts of $\mathcal{R}_1$ both converge, but to distinct limits.*

Proof: If some U_N from Theorem 7.5 is singular, then redefine the U_N product to start from some sufficiently large $n = n_0$. By the perturbation Theorem 6.1, and Theorem 7.5, the product Z_N of (4.2) thus converges to a finite matrix, and Theorem 4.1 applies. □

Via Corollary 7.6—and previous results for the $a^2 = b^2$ cases—we finally resolve the convergence domain for the Ramanujan AGM fraction, in the sense that the set equality $\mathcal{D}_1 = \mathcal{D}_0$ indeed holds. Note that, in terms of recurrence relations, we have hereby shown that (v_n) as defined in (1.4) converges to finite, bifurcated values (as n be even/odd) for the a, b parameters of Corollary 7.6, and in this way we also have immediate, precise knowledge of the asymptotic—and sometimes peculiar—behavior of sequences (t_n), (r_n), (q_n) for such parameter choices.

8 Matrix analysis toward a generalization of $\mathcal{R}_1$

Our exponential-sum resolution of the convergence domain was seen to be rather intricate, but it did, after all, provide explicit bounds on matrix elements, and did reveal some interesting aspects of multiple-Lerch sums. But there is a way to proceed more generally, to assail more general recurrence schemes than those attendant on the Ramanujan fraction. We hereby establish

Theorem 8.1. *Denote*

$$\Omega_n = \begin{bmatrix} 0 & \omega^n \\ \omega^{-n} & 0 \end{bmatrix}$$

where $|\omega| = 1, \omega \neq 1$. If z is an arbitrary complex number and a real, decreasing sequence (m_j) has $\sum_{j>0} m_j^2 < \infty$, then the matrix product

$$\prod_{j=1}^{n}(I + zm_j\Omega_j)$$

converges to a finite matrix as $n \to \infty$

Remark. Note that Theorem 8.1 implies the convergence in Theorem 7.5, since $\sum 1/n^2$ is finite. We have already intimated, though, as to the benefits of our quantitative multiple-Lerch sum analysis that led to Theorem 7.5.

In order to prove Theorem 8.1, we first establish some nomenclature and lemmas. Let (a_n) be a sequence of complex numbers, and let (ε_n) be a sequence of positive numbers. Denote the limit of (a_n) by a_∞. We write

$$(a_n) \prec (\varepsilon_n)$$

to mean that (a_n) converges with the caveat

$$|a_k - a_\infty| = O(\varepsilon_k).$$

Lemma 8.2. *Let (a_n) and (b_n) be complex sequences, let (ε_n) be a positive sequence, and let c be a complex number. Suppose that*

$$(a_n) \prec (\varepsilon_n) \text{ and } (b_n) \prec (\varepsilon_n).$$

Then

$$(a_n + b_n) \prec (\varepsilon_n), \quad (a_n b_n) \prec (\varepsilon_n), \quad (ca_n) \prec (\varepsilon_n).$$

Proof: The proof is straightforward. □

Lemma 8.3. *Let* (p_n) *be a decreasing sequence of non-negative real numbers that converges to* 0. *Then*

$$\left(\sum_{j=1}^{n} p_j w^j\right) \prec (p_n),$$

and

$$\left(\sum_{j=1}^{n} p_j \omega^{-j}\right) \prec (p_n).$$

Proof: By [16, Theorem 2.2 chapter 1] it follows that

$$\left|\sum_{j=n}^{\infty} p_j \omega^j\right| \le p_n \max_{k \ge n} \left|\sum_{j=n}^{k} \omega^j\right| \le \frac{2p_n}{|1-\omega|}.$$

□

Lemma 8.4. *The product*

$$\prod_{j=1}^{n} (1 + z m_j \omega^j)$$

converges as $n \to \infty$.

Proof: It follows from [9, p. 225] that if (a_n) is sequence such that $\sum_{j=1}^{n} a_j$ converges as $n \to \infty$, and $\sum_{j=1}^{n} |a_j|^2$ likewise converges, then $\prod_{j=1}^{n}(1 + a_j)$ also converges. Hence Lemma 8.4 follows from Lemma 8.3. □

Lemma 8.5. *Let*

$$U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

For all $n \in Z^+$, *let*

$$\Pi_U^n = \prod_{j=1}^{n} (I + z m_j \omega^j U) = \prod_{j=1}^{n} \begin{bmatrix} 1 & z m_j \omega^j \\ 0 & 1 \end{bmatrix},$$

$$\Pi_L^n = \prod_{j=1}^n (I + zm_j\omega^{-j}L) = \prod_{j=1}^n \begin{bmatrix} 1 & 0 \\ zm_j\omega^{-j} & 1 \end{bmatrix},$$

$$\Sigma_U^n = z\sum_{j=1}^n m_j\omega^j, \quad \Sigma_L^n = z\sum_{j=1}^n m_j\omega^{-j}.$$

Then (Π_U^n) *and* (Π_L^n) *converge, and*

$$\left(\Sigma_U^n\right) \prec (m_n), \quad \left(\Sigma_L^n\right) \prec (m_n). \tag{8.1}$$

Proof: We see by induction that

$$\Pi_U^n = \begin{bmatrix} 1 & \Sigma_U^n \\ 0 & 1 \end{bmatrix} \text{and } \Pi_L^n = \begin{bmatrix} 1 & 0 \\ \Sigma_L^n & 1 \end{bmatrix}.$$

Conditions (8.1) hold by Lemmas 8.2 and 8.3, so (Π_L^n) and (Π_U^n) converge. □

Lemma 8.6. *For all* $n \in Z^+$, *let*

$$\Pi_{UL}^n = \prod_{j=1}^n ((I + zm_j\omega^j U)(I + zm_j\omega^{-j}L)).$$

Then

$$\Pi_{UL}^n = \Pi_U^n \Pi_L^n \prod_{j=1}^n (I + R_j),$$

where

$$R_n = zm_n\omega^{-n}\begin{bmatrix} -\Sigma_U^{n-1} - \left(\Sigma_U^{n-1}\right)^2\Sigma_L^{n-1} & -\left(\Sigma_U^{n-1}\right)^2 \\ \Sigma_U^{n-1}\Sigma_L^n + \Sigma_U^{n-1}\Sigma_L^{n-1} + \Sigma_L^{n-1}\Sigma_L^n\left(\Sigma_U^{n-1}\right)^2 & \Sigma_U^{n-1} + \left(\Sigma_U^{n-1}\right)^2 - \Sigma_L^n \end{bmatrix}.$$

Here we interpret Σ_U^0 *and* Σ_L^0 *to be zero, so* $R_1 := 0$.

Proof: The proof is by induction on n. For $n = 1$ the result is clear. Suppose the result holds for n. Then

$$\begin{aligned} \Pi_{UL}^{n+1} &= (I + zm_{n+1}\omega^{n+1}U)\big(I + zm_{n+1}\omega^{-(n+1)}L\big)\Pi_{UL}^n \\ &= (I + zm_{n+1}\omega^{n+1}U)\big(I + zm_{n+1}\omega^{-(n+1)}L\big)\Pi_U^n\Pi_L^n\prod_{j=1}^n (I + R_j). \end{aligned}$$

We calculate the commutator

$$
\begin{aligned}
K &= \left[\left(I + zm_{n+1}\omega^{-(n+1)}L\right), \Pi_U^n\right] \\
&= \left[zm_{n+1}\omega^{-(n+1)}L, I + \Sigma_U^n U\right] = zm_{n+1}\omega^{-(n+1)}\left[L, \Sigma_U^n U\right] \\
&= zm_{n+1}\omega^{-(n+1)}\Sigma_U^n [L, U] \\
&= zm_{n+1}\omega^{-(n+1)} \begin{bmatrix} -\Sigma_U^n & 0 \\ 0 & \Sigma_U^n \end{bmatrix}.
\end{aligned}
$$

We have used the fact that $[L, U] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Since $XY = YX + [X, Y]$ for any 2×2 matrices X, Y, we have

$$
\begin{aligned}
\Pi_{UL}^{n+1} &= (I + zm_{n+1}\omega^{n+1}U)\left(\Pi_U^n\left(I + zm_{n+1}\omega^{-(n+1)}L\right) + K\right)\Pi_L^n \prod_{j=1}^{n}(I + R_j) \\
&= \Pi_U^{n+1}\Pi_L^{n+1}(I + V_n)\prod_{j=1}^{n}(I + R_j)
\end{aligned}
$$

where

$$
\begin{aligned}
V_n &= \left(\Pi_L^{n+1}\right)^{-1}\left(\Pi_U^n\right)^{-1} K \Pi_L^n \\
&= zm_{n+1}\omega^{-(n+1)} \begin{bmatrix} 1 & 0 \\ -\Sigma_L^{n+1} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\Sigma_U^n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\Sigma_U^n & 0 \\ 0 & \Sigma_U^n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Sigma_L^n & 1 \end{bmatrix} \\
&= zm_{n+1}\omega^{-(n+1)} \begin{bmatrix} -\Sigma_U^n - \left(\Sigma_U^n\right)^2 \Sigma_L^n & -\left(\Sigma_U^n\right)^2 \\ \Sigma_U^n \Sigma_L^{n+1} + \Sigma_U^n \Sigma_L^n + \Sigma_L^n \Sigma_L^{n+1}\left(\Sigma_U^n\right)^2 & \Sigma_U^n + \left(\Sigma_U^n\right)^2 \Sigma_L^{n+1} \end{bmatrix} \\
&= R_{n+1}.
\end{aligned}
$$

It follows that

$$
\Pi_{UL}^{n+1} = \Pi_U^{n+1}\Pi_L^{n+1}(I + R_{n+1})\prod_{j=1}^{n}(1 + R_j) = \Pi_U^{n+1}\Pi_L^{n+1}\prod_{j=1}^{n+1}(I + R_j).
$$

This completes the proof of Lemma 8.6. $\square$

Lemma 8.7. *The sequence* (Π_{UL}^n) *converges.*

Proof: By Lemmas 8.5 and 8.6, it is sufficient to show that $\prod_{j=1}^{n}(I+R_j)$ converges. From the definition of R_n and Lemmas 8.2 and 8.5, we see that

$$R_n = zm_n\omega^{-n}\begin{bmatrix} q_{11}^n & q_{12}^n \\ q_{21}^n & q_{22}^n \end{bmatrix}$$

where

$$\left(q_{ij}^n\right) \prec (m_n) \text{ for all } i \text{ and } j.$$

Thus

$$R_n = P_n + zm_n\omega^{-n}T$$

where

$$P_n := zm_n\omega^{-n}\begin{bmatrix} q_{11}^n - q_{11}^\infty & q_{12}^n - q_{12}^\infty \\ q_{21}^n - q_{21}^\infty & q_{22}^n - q_{22}^\infty \end{bmatrix} \quad \text{and} \quad T := \begin{bmatrix} q_{11}^\infty & q_{12}^\infty \\ q_{21}^\infty & q_{22}^\infty \end{bmatrix}.$$

We have a matrix norm relation

$$|P_n| = O\left(m_n^2\right), \quad \text{and} \quad \text{hence} \quad \sum_{n=1}^{\infty}|P_n| \text{ converges.}$$

By our perturbation Theorem 6.1, if we show that $\prod_{j=1}^{n}(I+zm_j\omega^{-j}T)$ converges, it will follow that $\prod_{j=1}^{n}(I+R_j)$ converges. From the development for V_n above, we see that

$$\det(T) = -\left(\Sigma_U^\infty\right)^2 \quad \text{and that} \quad \operatorname{trace}(T) = 0.$$

Hence the characteristic polynomial of T is given by

$$p(x) = x^2 - \left(\Sigma_U^\infty\right)^2.$$

If $\Sigma_U^\infty \neq 0$, then T has distinct eigenvalues $\pm\lambda$, so T is diagonalizable and there is a matrix M such that

$$I + zm_n\omega^{-n}T = M\begin{bmatrix} (1+\lambda zm_n\omega^{-n}) & 0 \\ 0 & (1-\lambda zm_n\omega^{-n}) \end{bmatrix}M^{-1},$$

and hence

$$\prod_{j=1}^{n}(I+zm_j\omega^{-j}T) = M\begin{bmatrix} \prod_{j=1}^{n}(1+\lambda zm_j\omega^{-j}) & 0 \\ 0 & \prod_{j=1}^{n}(1-\lambda zm_j\omega^{-j}) \end{bmatrix}M^{-1}.$$

This product converges by Lemma 8.4. If $\Sigma_U^\infty = 0$, then $T = 0$, so $\prod_{j=1}^n (I + zm_j\, \omega^{-j} T)$ certainly converges. □

We may now move to

Proof of the Theorem 8.1: We have

$$I + zm_j \Omega^j = (I + zm_j \omega^j U)(I + zm_j \omega^{-j} L) - z^2 m_n^2 UL.$$

By Lemma 8.7, we know that $\prod_{j=1}^n (I + zm_j \omega^j U)(I + zm_j \omega^{-j} L)$ converges, and since $\sum_{j=1}^n |z^{2j} m_j^2 UL| < \infty$, it follows from our perturbation Theorem 6.1 that $\prod_{j=1}^n (I + zm_j \Omega^j)$ converges. □

Let us now generalize the Ramanujan AGM fraction in the following manner. Define for an extra parameter c the continued fraction (which may or may not converge)

$$\mathcal{Q}\,(a, b, c) := \cfrac{1^c b^2}{1+\cfrac{2^c a^2}{1+\cfrac{3^c b^2}{1+\cfrac{4^c a^2}{1+\ddots}}}}$$

Note that $\mathcal{S}(a, b) = \mathcal{Q}(a, b, 2)$ so that the Ramanujan fraction (1.1) itself is the specific assignment

$$\mathcal{R}_1(a, b) = \frac{a}{1 + \mathcal{Q}(a, b, 2)}.$$

Thus we know, for example, that

$$\mathcal{Q}(1, 1, 2) = \frac{1}{\log 2} - 1.$$

With a view to our general Theorem 8.1, we may define a sequence $(v_n^{(c)})$ as a modification of our (v_n) in (1.4):

$$v_n^{(c)} := \frac{q_n}{\Gamma^{c/2}(n + 3/2)\kappa_n^{(n+1)/2}},$$

where as usual $\kappa_n := a^2, b^2$ as n be even, odd respectively. Now the algebra that opened Section 4 reads

$$\begin{bmatrix} v_{2n}^{(c)} \\ v_{2n-1}^{(c)} \end{bmatrix} = Y_n^{(c)} \begin{bmatrix} v_{2n-2}^{(c)} \\ v_{2n-3}^{(c)} \end{bmatrix},$$

where the Y matrix here is

$$Y_n^{(c)} := \begin{bmatrix} \frac{4n^c+1/a^2}{(4n^2-1/4)^{c/2}} & \frac{\omega^n}{a(2n+1/2)^{c/2}}\left(\frac{(2n-1)^2}{(2n-1)^2-1/4}\right)^{c/2} \\ \frac{\omega^{-n}}{a(2n-1/2)^{c/2}} & \left(\frac{(2n-1)^2}{(2n-1)^2-1/4}\right)^{c/2} \end{bmatrix}.$$

Thus the analogue of (4.2), namely

$$Z_N^{(c)} := \prod_{n=1}^{N} Y_n^{(c)}$$

can be written, when $|\omega := b^2/a^2| = 1$,

$$Z_N^{(c)} = \prod_{n=1}^{N}\left(I + \frac{1}{a(2n)^{c/2}}\Omega_n + O\left(\frac{1}{n^2}\right)\right).$$

Note also, by analogy with the determinantal development prior to (4.3) we have

$$\det\left(Z_\infty^{(c)}\right) = \left(\frac{\pi}{2}\right)^{c/2}.$$

Theorem 8.8. *For parameter choices $a \neq 0$, $|\omega := b^2/a^2| = 1$, $\omega \neq 1$, the generalized Ramanujan fraction*

$$\mathcal{Q}(a,b,c) := \cfrac{1^c b^2}{1+\cfrac{2^c a^2}{1+\cfrac{3^c b^2}{1+\cfrac{4^c a^2}{1+\ddots}}}}$$

diverges for any real $c > 1$, with the even/odd parts of $\mathcal{Q}$ converging to separate limits.

Proof: We may with impunity discard outright the $O(1/n^2)$ perturbation in $Z_N^{(c)}$ by Theorem 6.1, so taking $m_j := j^{-c/2}$ in Theorem 8.1, and employing Theorem 4.1 as before (except with the determinant $\pi/2 \to (\pi/2)^{c/2}$ throughout) we obtain convergence of the even/odd parts to separate values, hence divergence. $\square$

In fact, the bound of (1.5) for the separation of convergents in terms of v_n is to be modified simply by replacing v_n with $v_n^{(c)}$.

It could be that the precise convergence region for $\mathcal{Q}(a,b,c)$ for every real $c > 1$ is identical to that (region $\mathcal{D}_1$) of the Ramanujan case ($c := 2$); we have not investigated this, except to give Theorem 8.8 above. It is interesting that even our general convergence Theorem 8.1 does not directly handle the divergence question for $\mathcal{Q}(a,b,1)$.

We do know some exact evaluations for converging instances with $c = 1$, for example

$$\mathcal{Q}\,(a, a, 1) := \cfrac{1 \cdot a^2}{1+ \cfrac{2 \cdot a^2}{1+ \cfrac{3 \cdot a^2}{1+ \cfrac{4 \cdot a^2}{1+ \ddots}}}}$$

$$= \frac{4}{a e^{1/(2a^2)} (2\pi)^{1/2} \operatorname{erfc}(1/(a\sqrt{2}))} - 1,$$

where erfc is the standard complementary error function, $\operatorname{erfc}(z) := (2/\sqrt{\pi}) \int_z^\infty e^{-t^2} dt$. Perhaps surprisingly, there is another one-parameter class of exact evaluations, namely

$$\mathcal{Q}\left(\sqrt{1+z^2}, z, 1\right) = \frac{z}{\arctan z} - 1,$$

valid at least for $\operatorname{Re}(z^2) > -1/2$ [11, 3.6.10, p. 571].

Consistent with the scope of Theorem 8.8, it appears that there are pairs a, b with $a/b \neq 1$, $|a/b| = 1$ such that $\mathcal{Q}(a, b, 1)$ actually *converges*. Using the above arctan formula, for example, we have, at least formally,

$$\mathcal{Q}\left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 1\right) = \frac{i}{\sqrt{2}\arctan(i/\sqrt{2})} - 1 = -0.1977218382755\ldots,$$

and numerical experiments show that the continued fraction appears to approach this value (which we say because $\operatorname{Re}(z^2) = 1/2$ in this case, which is outside the validity range given—indeed the region of validity for the arctan representation must certainly be wider than has been proven). These observations suggest that research is in order on the evidently discontinuous change in convergence behavior at real $c = 1$.

There are, of course, other directions in which to generalize the Ramanujan construct, one of the most alluring being extensions of the defining set of recurrence relations, such as the relations relevant to the construct

$$\cfrac{1^2 c^2}{1+ \cfrac{2^2 b^2}{1+ \cfrac{3^2 a^2}{1+ \cfrac{4^2 c^2}{1+ \ddots}}}}$$

that is, the n-th numerator is $n^2 \cdot (c^2, b^2, a^2)$ as $n = (1, 2, 0)$ modulo 3, so there are three essential relations (in the sense that the recurrence in our Abstract amounts to two essential relations).

Such a generalization would naturally lead into 3×3 matrix theory (so that Theorem 6.1 is intact, but such as Theorems 7.1 and 8.1 need be carefully extended). It could be that some nontrivial analogue of the "chaotic" convergents' behavior for the case $(a, b) = (i, i)$ of the Ramanujan fraction exists for the above construct—perhaps for some choice other than $(a, b, c) = (i, i, i)$—and such a finding could conceivably lead to a new class of chaotic generators.

9 The classical Pincherle and Auric theorems

The Pincherle theorem [11, Theorem 7, p. 202] says essentially that if *any* two solutions (with neither being the zero sequence $(0, 0, \ldots)$) to a given recurrence have the same asymptotic behavior (in a quantifiable sense), then the associated continued fraction diverges. Moreover, the converse is true: If one can find two nonzero recurrence solutions with one dominating the other (say $t_n(t_0, t_i)/t_n(t_0', t_1') \to 0$, in which case the numerator sequence is called *minimal*), then the fraction converges. The Auric theorem [11, Theorem 10, p. 207] gives a convergence criterion (and an actual fraction value) based on a minimal recurrence solution. With a view to the Auric-Pincherle theory, one may construct an entity relevant to the (v_n) matrix iteration (4.1):

$$\frac{1}{a} \sum_{n=0}^{\infty} \frac{(-1)^n}{\bar{v}_n \bar{v}_{n-1}}, \tag{9.1}$$

where we use the notation $\bar{v}$ to signify that we are allowed to adopt initial conditions other than the standard ones in (4.4). (Note that (9.1) is equivalent to the literature presentation [11, p. 202] of this key sum.) Now (by Pincherle) if initial conditions $\bar{v}_0, \bar{v}_1$ can be found such that this sum (9.1) diverges to ∞, then $(\bar{v}_n)$ is a minimal solution of the recurrence. So the continued fraction associated with $(\bar{v}_n)$ converges if and only if (9.1) diverges to ∞. Similarly (by Auric) if (9.1) diverges to ∞ then, remarkably, the actual *value* of the continued fraction can be given in terms of the initial conditions, as $\mathcal{S}(a, b) = -a\bar{v}_0/(2\bar{v}_1)$. Note an immediate echo of one of our previous results: If the matrix Z_∞ of (4.3) exists, then (9.1) cannot diverge to ∞ (for any initial conditions on $\bar{v}$) and so the associated fraction must diverge.

Conversely, there is point of consistency: Closed-form values for *convergent* fractions in the Auric theory are entirely consistent with the developments in our Sections 2 and 3. For example, when $a = b$ with Re $(a) > 0$ it turns out that a minimal solution can be written down exactly, as

$$\bar{v}_n := \frac{\Gamma(n+2)F_{n+1}(-a)}{\Gamma(n+3/2)a^{n+1}}$$

(equivalently, $\bar{t}_n := F_n(-a)$ for the relevant $(\bar{t}_n)$ sequence, with F_n as in (2,3)) so that (9.1) is seen via (2.6) to diverge to complex infinity, and sure enough the fraction value $\mathcal{S}(a, a) = -a\bar{v}_0/(2\bar{v}_1)$ agrees with the hypergeometric-ratio term of (3.2).

These and some other classical ideas are mirrored in our matrix analysis; indeed, we have shown along the way that literature analyses such as Jacobsen-Masson divergence theory—which established divergence of $\mathcal{S}(i, i)$ in [4]—are also consistent with our present development. We hope that our methods can add to the already voluminous theory of complex continued fractions. It would be interesting to forge a comprehensive theory—beyond, say, the particular case of the Ramanujan AGM fraction—that fuses all of these matrix methods with the classical approaches.

10 Some dynamics and their pictures

We may think of the capstone *dynamical system* $t_0 := t_1 := 1$:

$$t_n \hookleftarrow \frac{1}{n}\, t_{n-1} + \kappa_{n-1}\left(1 - \frac{1}{n}\right) t_{n-2}$$

where $\kappa_n = a^2, b^2$ for n even, odd respectively, of (1.2) as a *black box*. Numerically all one learns is that t_n is slowly tending to zero.

Pictorially, we learn significantly more (Fig. 1 shows the iterates swirling in towards zero, with the shading changing every few hundred iterates), and after scaling by $\sqrt{n}$, and on coloring odd and even iterates distinctly, fine structure appears—depending on whether a and/or b is a root of unity in the multiplicative group of the circle (Figs. 2 and 3 show four representative choices of iterates of $\sqrt{n}t_n$, in the Argand plane).

Indeed these pictures were part of our road to discovery. The behavior is now entirely explained by Corollary 7.6. Using (1.2) and (1.4) we have that

$$\sqrt{n+1}t_{n+1} \sim v_n \sqrt{\kappa_n}^{n+1}.$$

When $a^2/b^2 \neq 1$, $|a/b| = 1$ we know that v_n has bifurcated convergent subsequences while $\sqrt{\kappa_n}$ alternates between a and b, so an appeal to Weyl's uniform convergence criterion explains the dynamics.

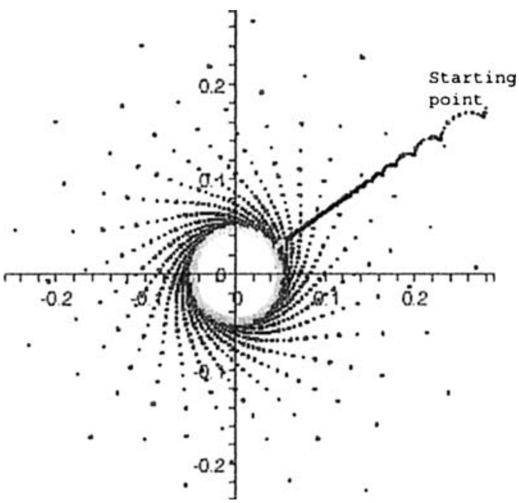

Fig. 1 The decay of t_n. Initial conditions t_o, t_1 are chosen, then a plot in the complex plane shows the amplitude $|t_n|$ decaying as $1/\sqrt{n}$

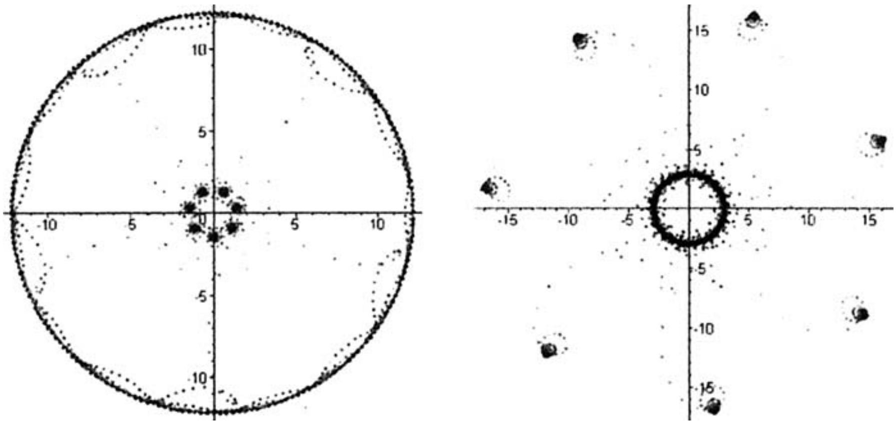

Fig. 2 The attractors for $|a| = |b| = 1$ with exactly one of a, b a root of unity. Plotted points are $\sqrt{n}t_n$, which we expect to be bounded

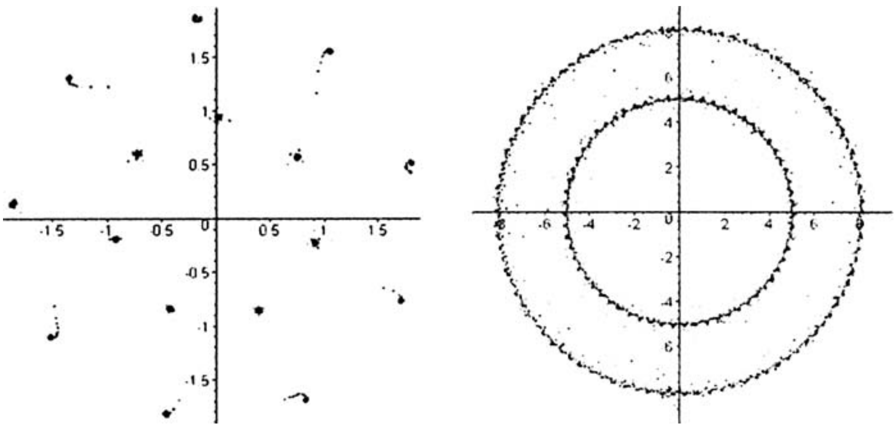

Fig. 3 The attractors for $|a| = |b| = 1$ with both or neither a, b being roots of unity. Plotted points are as in our previous figure

11 Some unresolved issues

We finish by listing some open questions.

- We again mention that we do not know a single nontrivial closed-form value of $\mathcal{R}_1(a, b)$; meaning, except for $a = 0$ or $b = 0$ or $a = b$ and some cases $\mathcal{R}_1(a, b) = \infty$ (see [4]) we have no closed forms. Could the minimal-solution scheme implicit in the Pincherle-Auric theory somehow reveal nontrivial closed forms?
- The dynamics in Figs. 2 and 3 are now explained, but what happens for more general, or even random choices of κ_n? Computational experiment suggests that (1.2) behaves similarly when the κ_n are chosen cyclically with any even period but not when chosen with an odd period. When κ_n is chosen randomly, the dynamics appear surprisingly deterministic. Why should the dynamics be so sensitive to the

parity (odd/even) of cycles, yet so robust when the parameters are randomly chosen? What is the nature of the convergence of t_n for cycles, odd or even, of length $c > 2$. For random κ_n, does the sequence t_n converge, almost surely or otherwise? Related results may be found in [5].

- Perhaps there is a universal, hypergeometric form for $\mathcal{Q}(a, b, 1)$? The existence of *two* one-parameter families of forms for $\mathcal{Q}(a, b, 1)$ suggests there may be more accessible forms.
- Could it be that the precise convergence domain $\mathcal{D}_0 = \mathcal{D}_1$ is also valid for the generalized Ramanujan fraction $\mathcal{Q}(a, b, c)$, any real $c > 1$? (Theorem 8.8 is consistent with this hypothesis.)
- We mentioned the other appealing generalization in which a, b, c as appearing modulo 3 or higher. Such would require 3×3 matrix theory, etc. Again computation suggests similar dynamics for yet other 3-fold variants of (1.2) such as

$$t_n = \frac{1}{n} t_{n-1} + \frac{1}{n} t_{n-2} - \frac{n-2}{n} t_{n-3}.$$

- There are at least potential cryptographic/chaos-generator applications, and likewise Monte-Carlo applications of our (t_n). In what sense is a *modular* iteration

$$t_n = (t_{n-1} - (n-1)t_{n-2}) \cdot n^{-1} \bmod N$$

chaotic, or at least useful? It is not inconceivable that such a chaotic scheme could apply to factoring (of N), much like the celebrated Pollard-rho factoring method.

Acknowledgments We thank B. Berndt, J. Buhler, J. Coope, D. Griffiths, W. Jones, L. Lorentzen, R. Luke, and S. Rayhawk for many valuable discussions.

References

1. Andrews, G., Askey, R., Roy, R.: Special Functions. Cambridge University Press (1999)
2. Borwein, J., Bailey, D., Girgensohn, G.: Experimentation in Mathematics: Computational Paths to Discovery. AK Peters (2004)
3. Borwein, J., Crandall, R., Fee, G.: On the Ramanujan AGM fraction. Part I: The real-parameter case. Experimental Mathematics, **13**, 275–286 (2004) [CECM Preprint 2003: 208/213]
4. Borwein, J., Crandall, R.: On the Ramanujan AGM fraction. Part II: The complex parameter case. Experimental Mathematics, **13**, 287–296 (2004) [CECM Preprint 2003: 214]
5. Borwein, J.M., Luke, D.R.: Dynamics of some random continued fractions, Abstract and Applied Analysis, 449–468 (2005) [CoLab Preprint 275]
6. Gill, J.: Infinite compositions of Mobius transforms. Trans. Amer. Math. Soc. **176**, 479–487 (1973)
7. Golub, G.H., van Loan, C.F.: Matrix Computations. The John Hopkins University Press (1989)
8. Jones, W., Thron, W.: Continued Fractions: Analytic Theory and Applications. Addison-Wesley (1980)
9. Knopp, K.: Theory and Application of Infinite Series. Dover (1990)
10. Luke, Y.: The special functions and their approximations, vol. I. In: Bellman, R. (ed.) Mathematics in science and engineering, vol. LIII. Academic Press, New York (1969)
11. Lorentzen, L., Waadeland, H.: Continued Fractions with Applications. NorthHolland (1992)
12. Megginson, R.E.: An Introduction to Banach Space Theory. Springer-Verlag (1998)
13. Olver, F.W.J.: Asymptotics and Special Functions. Academic Press (1974)
14. Petkovsek, M., Wilf, H., Zeilberger, D.: $A = B$, AK Peters (1996)
15. Wall, H.S.: Analytic Theory of Continued Fractions. D. Van Nostrand (1948)
16. Zygmund, A.: Trigonometric Series, vol I, Cambridge University Press (1959)