REFINED CONVEXITY AND SPECIAL CASES OF THE BLASCHKE-SANTALO INEQUALITY[1]

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Abstract. We derive the l_p version of the classical Blaschke-Santalo inequality for polar volumes as a consequence of more subtle convexity estimates for the volume of the $p-$ball in Euclidean space. We also give analogs for the (p, q)-substitution norms.

1 The problem

Take a convex body C in $\mathbb{R}^n$ and define its *polar body* to be the set

$$C^\circ \ := \ \{y \in \mathbb{R}^n : \langle y, x\rangle \leq 1 \text{ for all } x \in C\}.$$

Denoting the n-dimensional Euclidean volume of a set $S \subseteq \mathbb{R}^n$ by $V_n(S)$, the Blaschke-Santalo inequality says that

$$V_n(C)\, V_n(C^\circ) \ \leq \ V_n(E)\, V_n(E^\circ) \ = \ V_n(B_n(2))^2 \tag{1}$$

where E is any ellipsoid and $B_n(2)$ is the unit ball with respect to the Euclidean norm, see [6] or [5].

In this note, we investigate this inequality in the case of the unit ball with respect to the $\|\cdot\|_p$-norm in $\mathbb{R}^n$,

$$B_n(p) \ := \ \{x \in \mathbb{R}^n : \sum_{i=1}^{n} |x_i|^p \leq 1\}.$$

We have $B_n(p)^\circ = B_n(p')$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

Is there a direct proof of (1) when $C = B_n(p)$? What, in this special case, is the underlying reason for the inequality (1)? It will turn out that the volume function $V_n(p) \equiv V_n(B_n(p))$ satisfies a much more general set of inequalities which amounts to a modified form of log-convexity for this function. This modified convexity (with respect to two arbitrary means) is of the type investigated by J. Aczél in [1] and later by J. Matkowski and J. Rätz in [3] and [4].

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The volume of the unit ball in the $\|\cdot\|_p$-norm, $V_n(p)$, was first determined by Dirichlet by explicitly evaluating the iterated integrals. He obtained

$$V_n(p) \;=\; 2^n \frac{\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})},$$

cf. [2], Section 1.8. The following Maple code derives this formula as an iterated integral for arbitrary p and fixed n. The intermediate steps give beta function values. It can easily be converted into a human proof valid for arbitrary n.

```
vol := proc(n)
 local f,x,i,ul,u,j,t; global p;
 p := evaln(p);
 if n=1 then 2 else
  f := (1-add(x[i]^p,i=1..n-1))^(1/p);
  for i from n-1 by -1 to 1 do
   f := subs(x[i]=t,f); f := int(f,t);
   ul := 1-add(x[j]^p,j=1..i-1); u := ul^(1/p);
   f := subs(t^p=ul,f); f := subs(t=u,f);
   f := map(normal,f); f := simplify(f);
  od;
  2^n*f;
 fi; end:

# The volume of the p-ball in R^3 is: V3 := vol(3);
# The volume of the Euclidean sphere is : eval(V3,p=2);
```

Also, from Euler's product formula

$$\frac{1}{\Gamma(z)} \;=\; z\,e^{\gamma z} \prod_{k=1}^{\infty} \left[\left(1+\frac{z}{k}\right) e^{-z/k}\right]$$

we get the representation

$$V_n(p) \;=\; 2^n \prod_{k=1}^{\infty} \frac{1+\frac{n}{kp}}{\left(1+\frac{1}{kp}\right)^n}.$$

For $n=2$ and $n=3$, this reduces, quite elegantly, to

$$V_2(p) \;=\; 4\prod_{k=1}^{\infty}\left(1-\frac{1}{(pk+1)^2}\right) \quad \text{and} \quad V_3(p) \;=\; 8\prod_{k=1}^{\infty}\left(1-\frac{3pk+1}{(pk+1)^3}\right).$$

In particular, explicit evaluations of these products such as $V_2(4) = \frac{\pi^{3/2}}{\Gamma(3/4)^2}$ follow.

2 Generalized Convexity and the Main Results

Theorem. *For every $\alpha > 1$, the function* $V_\alpha(p) := 2^\alpha \dfrac{\Gamma(1+\frac{1}{p})^\alpha}{\Gamma(1+\frac{\alpha}{p})}$ *satisfies*

$$V_\alpha(p)^\lambda \, V_\alpha(q)^{1-\lambda} \;<\; V_\alpha\left(\frac{1}{\frac{\lambda}{p}+\frac{1-\lambda}{q}}\right), \tag{2}$$

where $p, q > 0$, $p \neq q$, and $\lambda \in (0,1)$.

It is trivial that inequality (2) can be iterated finitely to give

$$\prod_i V_\alpha(p_i)^{\lambda_i} \;<\; V_\alpha\left(\frac{1}{\sum_i \lambda_i/p_i}\right)$$

for all $\lambda_i \in (0,1)$ with $\sum_i \lambda_i = 1$, and all $p_i > 0$ with not all of them equal. In particular, when $\alpha = n$, $\lambda_1 = \lambda_2 = 1/2$ and $1/p_1 + 1/p_2 = 1$ we recover the $p-$norm case of the Blaschke-Santalo inequality (1).

We will now develop a proof for the theorem, with some digressions.

If we define $U_\alpha(p) := -\ln(V_\alpha(p)/2^\alpha)$, then inequality (2) is equivalent to

$$U_\alpha\left(\frac{1}{\frac{\lambda}{p}+\frac{1-\lambda}{q}}\right) \;<\; \lambda\, U_\alpha(p) + (1-\lambda)\, U_\alpha(q) \tag{3}$$

for the asserted values of p, q, λ. This is a modified form of convexity, where the (weighted) arithmetic mean in the argument of U_α is replaced by the (weighted) harmonic mean. This deserves closer attention, because such a modified convexity can be defined at least for more general quasi-arithmetic means. Thus, take a continuous, strictly monotonic function $\varphi : I \to \mathbb{R}$. Then

$$M(x,y) \;:=\; \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)$$

is called a *quasi-arithmetic mean*, $M : I^2 \to I$, and similarly,

$$M^{(\lambda)}(x,y) \;:=\; \varphi^{-1}(\lambda\, \varphi(x) + (1-\lambda)\varphi(y))$$

for $\lambda \in [0,1]$ is the weighted version of M. Here, $\lambda \in (0,1)$ and $x < y$ always implies $x < M(x,y) < y$. The function φ is called the *Kolmogoroff-Nagumo function* of M. Of special interest are the *power means* M_a on $\mathbb{R}_+$, defined by

$$\varphi_a(x) \;:=\; \begin{cases} x^a & \text{if } a \neq 0, \\ \ln(x) & \text{if } a = 0. \end{cases}$$

They satisfy $M_a(x,y) < M_b(x,y)$ for $a < b$ if $x \neq y$. For $a = 1$, we get the arithmetic mean $A = M_1$, for $a = 0$ the geometric mean $G = M_0$, and for $a = -1$ the harmonic mean $H = M_{-1}$.

For any two quasi-arithmetic means M, N (with Kolmogoroff-Nagumo functions φ, ψ defined on intervals I, J), a function $f : I \to J$ can be called (M,N)-*convex* if it satisfies

$$f(M^{(\lambda)}(x,y)) \;\leq\; N^{(\lambda)}(f(x), f(y)) \tag{4}$$

for all $x, y \in I$ and $\lambda \in [0,1]$, and *strictly* M-*convex* if the inequality is strict except for $x = y$ or $\lambda = 0, 1$. If N is the arithmetic mean, $N = A$, we just say that f is M-convex.

We remark that for power means M_a, M_b with $a < b$ we have the implications

$$\begin{aligned} f \ M_b\text{-convex and } f \text{ increasing} &\implies f \ M_a\text{-convex}, \\ f \ M_a\text{-convex and } f \text{ decreasing} &\implies f \ M_b\text{-convex}. \end{aligned}$$

Since for differentiable f, usual convexity gives rise to characterizations in terms of the derivatives of f, one may ask if the same is true for (M,N)-convexity. However, it turns out that things are much simpler than they appear on first glance. Assume for convenience that ψ is strictly increasing. Then simply set $s := \varphi(x)$ and $t := \varphi(y)$ in (4) to obtain after some manipulations the equivalent inequality

$$\psi\left(f\left(\varphi^{-1}\left(\lambda s + (1-\lambda)t\right)\right)\right) \;\leq\; \lambda\psi(f(\varphi^{-1}(s))) + (1-\lambda)\psi(f(\varphi^{-1}(t)))$$

for all $s, t \in \varphi(I)$. Thus f is M-convex on I if and only if $\psi \circ f \circ \varphi^{-1}$ is convex on $\varphi(I)$ in the usual sense. This was probably first observed by J. Aczél in [1]. In particular, if f and φ are differentiable with $\varphi'(x) \neq 0$ on I, then f is M-convex if and only if $(\varphi^{-1})'(x)\, f'(\varphi^{-1}(x))$ is increasing; f is strictly M-convex if the monotonicity is strict.

This concludes the brief excursion to the realm of modified convexity, except for the remark that one can, of course, consider an even wider notion of convexity by taking two strict means M, N (not necessarily quasi-arithmetic) and calling a function f (M,N)-*midpoint-convex* if it satisfies

$$f(M(x,y)) \;\leq\; N(f(x), f(y))$$

for $x, y \in I$. Such convexity appears to be more difficult than the special case discussed above, and it might be interesting to study this in more detail.

In this setting, log-convexity of a function is precisely (A,G)-convexity. For example, the Gamma function Γ is (A,G)-convex. As another example, the theorem says that $1/V_n$ is (H,G)-convex. Since $1/V_n$ is decreasing, it is also (A,G)-convex, and the arithmetic-geometric mean inequality implies that $1/V_n$ is both (H,A) and (A,A)-convex.

On the other hand, V_n itself is neither convex nor concave for $n \geq 3$. In fact, $V_3(p)$ has an inflection point at $p = 1.0823906..$, $V_4(p)$ at $p = 1.6369256..$, $V_5(p)$ at $p = 2.1925855..$, and these seem to increase with increasing n.

We can now apply these findings to the function U_α on $\mathbb{R}_+$ with M as the harmonic mean, i.e., M is generated by the function $\varphi(x) = 1/x$. We employ the *psi function*

$$\psi(x) \;:=\; (\ln\Gamma(x))',$$

and we note the identities

$$\psi'(1+x) \;=\; \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} \;=\; \int_0^{\infty} \frac{u}{e^u - 1} e^{-ux}\, du,$$

valid for all $x > 0$ (see [2]). In order to prove inequality (3) and thus the theorem, we must prove that the function

$$W_\alpha(x) \;:=\; U_\alpha(1/x) \;=\; \ln(\Gamma(1+\alpha x)) - \alpha \ln(\Gamma(1+x))$$

is strictly convex on $\mathbb{R}_+$ for every $\alpha > 1$. This is true if and only if $W''_\alpha(x) = \alpha^2 \psi'(1+\alpha x) - \alpha\, \psi'(1+x)$ is strictly positive on $\mathbb{R}_+$. Multiplying by x/α, we see that this is true if the function $x \mapsto x\, \psi'(1+x)$ is strictly increasing on $\mathbb{R}_+$. This is what we will now show to conclude the proof of the theorem. In fact, we shall show more.

Lemma. *The function* $\rho(x) := \dfrac{d}{dx} x\psi'(1+x)$ *is completely monotonic on* $[0, \infty)$, *i.e.,* $(-1)^k \rho^{(k)}(x) \geq 0$ *for* $x > 0$ *and* $k = 0, 1, 2, \ldots$.

Proof. We have that, for $x > 0$,

$$\psi'(1+x) \;=\; \int_0^{\infty} f(u) e^{-ux}\, du, \qquad \text{where } f(u) \;:=\; \frac{u}{e^u - 1},$$

so that

$$\rho(x) \;=\; \int_0^{\infty} f(u) e^{-ux}\, du - x \int_0^{\infty} u f(u) e^{-ux}\, du.$$

Integrating the final integral by parts yields

$$x \int_0^{\infty} u f(u) e^{-ux}\, du \;=\; \int_0^{\infty} \{f(u) + u f'(u)\} e^{-ux}\, du.$$

Hence, for $x > 0$,

$$\rho(x) \;=\; -\int_0^{\infty} u f'(u) e^{-ux}\, du,$$

and consequently

$$\rho^{(k)}(x) \;=\; -(-1)^k \int_0^{\infty} u^{k+1} f'(u) e^{-ux}\, du \qquad \text{for } k = 1, 2, \ldots.$$

Since

$$-f'(u) \;=\; \frac{(u-1)e^u + 1}{(e^u - 1)^2} \;>\; 0 \qquad \text{for } u > 0,$$

it follows that $\rho(x)$ is completely monotonic on $[0, \infty)$. A detailed study of completely monotonic functions can be found in [7]. ☺

3 Comments and Extensions

1. We remark that we can also give a positive lower bound on $(\frac{2}{3}, \infty)$ for the function $\rho(x)$, namely

$$0 \;<\; \frac{1}{2x^2} - \frac{1}{3x^3} \;\le\; \rho(x) \qquad \text{for all } x > \frac{2}{3}. \tag{5}$$

To prove (5), we use a variant of the formula on p. 29 of [2] for $n = 1$,

$$\begin{aligned}\ln\Gamma(1+x) \;&=\; \left(x + \frac{1}{2}\right)\ln(x) - x + \frac{1}{2}\ln 2\pi + \frac{1}{12x} \\ &\qquad - \frac{2}{x}\int_0^\infty \left(\int_0^t \frac{z^2}{x^2+z^2}\,dz\right)\frac{dt}{e^{2\pi t}-1}.\end{aligned} \tag{6}$$

For notational convenience, we define an operator L by $(Lg)(x) := g''(x) + x\,g'''(x)$ and note that $\rho(x) = L(\ln\Gamma)(1+x)$. Now

$$L(\text{first line of (6)}) \;=\; \frac{1}{2x^2} - \frac{1}{3x^3},$$

and

$$L\left(\frac{1}{x\,(x^2+z^2)}\right) \;=\; -4\,\frac{12x^6 + 3x^4z^2 + 4x^2z^4 + z^6}{x^3\,(x^2+z^2)^4},$$

which shows that L applied to the second line of (6) is positive. Thus

$$\rho(x) \;=\; L(\ln\Gamma)(1+x) \;\ge\; \frac{1}{2x^2} - \frac{1}{3x^3}.$$

2. Now one might ask the question of whether the *surface* area $S_n(p)$ of the ℓ^p-ball in $\mathbb{R}^n$ satisfies similar convexity conditions. We do not know the answer. The area $S_n(p)$ in general does not have an explicit representation; it can, however, be expressed via the integral

$$S_n(p) \;=\; 2^n \int_{B_{n-1}(p)} \left(1 + \frac{\sum_{i=1}^{n-1} x_i^{2p-2}}{\left(1 - \sum_{i=1}^{n-1} x_i^p\right)^{(2p-2)/p}}\right)^{1/2} dx_1 \ldots dx_{n-1}.$$

Evaluating this integral numerically for $n = 2$, it seems that neither $\log(S_2(p))$ nor $\log(S_2(1/p))$ are convex or concave on $[1, 2]$, but that S_2 itself is concave on $[2, \infty]$.

3. A most useful generalization of the ℓ^p-norms are the *substitution norms*:

$$\|(x_1, x_2, \cdots, x_n)\|_{p,\mathbf{q}} := \|(\|x_1\|_{q_1}, \|x_2\|_{q_2}, \cdots, \|x_n\|_{q_n})\|_p$$

where $\mathbf{q} = (q_1, \ldots, q_n)$ and where each x_i lies in some Euclidean space $\mathbb{R}^{m_i}$. Denoting by $V_{n,\mathbf{m}}(p, \mathbf{q})$ the volume of the unit ball $B_{n,\mathbf{m}}(p, \mathbf{q})$ with respect to this norm (with $\mathbf{m} = (m_1, \ldots, m_n)$), we again have a closed form:

$$V_{n,\mathbf{m}}(p,\mathbf{q}) \;=\; V_{\sum m_i}(p) \cdot \prod_{i=1}^n \frac{V_{m_i}(q_i)}{V_{m_i}(p)} \;=\; \frac{\prod_{i=1}^n \Gamma(1 + \frac{m_i}{p})}{\Gamma(1 + \frac{1}{p}\sum_{i=1}^n m_i)} \cdot \prod_{i=1}^n V_{m_i}(q_i). \tag{7}$$

If $q_i \equiv q$ and $m_i \equiv m$, this reduces to

$$V_{n,m}(p,q) \;=\; V_{nm}(p)\left(\frac{V_m(q)}{V_m(p)}\right)^n \;=\; V_n\left(\frac{p}{m}\right)\left(\frac{V_m(q)}{2}\right)^n,$$

which exhibits hidden symmetries and pretty special cases. The simplest case is that of the volume $V_{n,\mathbb{C}}(p)$ of the ℓ^p-ball in $\mathbb{C}^n$ where all q_i and all m_i equal 2.

To prove (7), we first remark the identity

$$\int_{\|x\|_q \le 1} \left(1 - \|x\|_q^p\right)^{\mu/p} dx \;=\; 2^m \frac{\Gamma(1+\frac{1}{q})^m}{\Gamma(1+\frac{m}{q})} \cdot \frac{\Gamma(1+\frac{\mu}{p})\,\Gamma(1+\frac{m}{p})}{\Gamma(1+\frac{\mu+m}{p})} \tag{8}$$

for all $\mu \ge 0$, which follows by restricting the integration to the positive orthant and then using Theorem 1.8.5 in [2] with $f(t) := (1-t^{p/q})^{\mu/p}$. The resulting integral in that theorem can be reduced to a beta function.

The volume formula (7) follows from this by induction over n. In fact, using homogeneity of volume and Fubini's theorem, the volumes are easily seen to satisfy

$$V_{n,\mathbf{m}}(p,\mathbf{q}) \;=\; \int_{\|x_n\|_{q_n} \le 1} \left(1 - \|x\|_{q_n}^p\right)^{\sum_{i=1}^{n-1} m_i/p} dx \cdot V_{n-1,\mathbf{m}^*}(p,\mathbf{q}^*) \tag{9}$$

where $\mathbf{a}^* = (a_1,\ldots,a_{n-1})$ for a vector $\mathbf{a} = (a_1,\ldots,a_n)$. Now, (7) is true for $n=1$, because then the substitution norm is just the q_1-norm in $\mathbb{R}^{m_1}$, and the induction step follows by (8) and (9).

To extend and recover the Blaschke-Santalo inequality in this case, we note that $B_{n,\mathbf{m}}(p,\mathbf{q})^\circ = B_{n,\mathbf{m}}(p',\mathbf{q}')$ where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q_i}+\frac{1}{q_i'}=1$. Similarly to the considerations in Section 2, the inequality follows if we prove that *the function* $1/V_{n,\mathbf{m}}(p,\mathbf{q})$ *is* (H,G)*-convex in every component.*

Since the volume (7) depends multiplicatively on the p- and the q-components, and since (H,G)-convexity of the q-components is precisely the content of our main theorem, all that remains to be proved is that the function

$$W_{n,\mathbf{m}}(x) \;:=\; \ln\Gamma\left(1 + x\sum_{i=1}^n m_i\right) - \sum_{i=1}^n \ln\Gamma(1 + x\,m_i)$$

is strictly convex in x on $\mathbb{R}_+$ for all vectors $\mathbf{m}$ of positive reals. This is true if and only if

$$W''_{n,\mathbf{m}}(x) \;=\; \left(\sum_{i=1}^n m_i\right)^2 \psi'\left(1 + x\sum_{i=1}^n m_i\right) - \sum_{i=1}^n m_i^2\,\psi'(1 + x\,m_i)$$

is strictly positive on $\mathbb{R}_+$. But this again follows from the strict monotonicity of the function $x \mapsto x\,\psi'(1+x)$, because we can estimate

$$m_i \cdot \psi'(1 + x\,m_i) \;<\; \sum m_i \cdot \psi'(1 + x\sum m_i).$$

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