ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 3, Fall 2001

ONE-SIDED TAUBERIAN THEOREMS FOR DIRICHLET SERIES METHODS OF SUMMABILITY

DAVID BORWEIN, WERNER KRATZ AND ULRICH STADTMÜLLER

ABSTRACT. We extend recently established two-sided or O-Tauberian results concerning the summability method $D_{\lambda,a}$ based on the Dirichlet series $\sum a_n e^{-\lambda_n x}$ to one-sided Tauberian results. More precisely, we formulate one-sided Tauberian conditions, under which $D_{\lambda,a}$-summability implies convergence. Our theorems contain various known results on power series methods of summability and, in the so-called high index case we even obtain a new result for such methods. Our method of proof uses asymptotic properties of the Dirichlet series subject to the assumption that a_n and λ_n can be interpolated by smooth functions. In addition we develop refined Vijayaraghavan-type results which enable us to infer the boundedness of sequences from the boundedness of their $D_{\lambda,a}$-means and the one-sided Tauberian conditions.

1. Introduction and main results. Suppose throughout that $\{\lambda_n\}$ is an unbounded and strictly increasing sequence of positive numbers, that $\{a_n\}$ is a sequence of nonnegative numbers, and that the Dirichlet series

$$a(x) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}$$

has abscissa of convergence $\sigma \in [-\infty, \infty)$. Let $\{s_n\}$ be a sequence of *real* numbers. The Dirichlet series summability method $D_{\lambda,a}$ is defined as follows:

$$s_n \to s\,(D_{\lambda,a}) \quad \{\text{or } s_n = O(1)(D_{\lambda,a})\}$$

$$\text{if} \quad \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \quad \text{is convergent for} \quad x > \sigma, \quad \text{and}$$

$$\sigma(x) := \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \to s \quad \{\text{or } \sigma(x) = O(1)\} \quad \text{as } x \to \sigma+$$

Research supported in part by the Natural Sciences and Engineering Research Council of Canada.
1991 AMS *Mathematics Subject Classification.* 40G10, 40E05.
Key words and phrases. Tauberian, Dirichlet series methods.
Received by the editors on April 20, 1999, and in revised form on May 22, 2000.

through real values. It is well known (and easy to verify) that if $a(x) \to \infty$ as $x \to \sigma+$ and if $a_n \neq 0$ for infinitely many n, then $D_{\lambda,a}$ is regular (which will always be the case in our results), i.e., $s_n \to s$ implies $s_n \to s\,(D_{\lambda,a})$.

Let the real functions g and λ satisfy the following conditions:
(C)
$$\begin{cases} g, \lambda \in C_2[x_0, \infty) \text{ for some } x_0 \in \mathbf{N}, \\ \lambda(x_0) > 0 \text{ and } \lambda(x) \to \infty \text{ as } x \to \infty, \\ \lambda'(x) > 0 \text{ and } \left(\dfrac{g'(x)}{\lambda'(x)}\right)' > 0 \text{ on } [x_0, \infty), \\ \dfrac{\lambda'(x)}{\lambda(x)} \text{ is nonincreasing, and } \dfrac{x\lambda'(x)}{\lambda(x)} \text{ is nondecreasing on } [x_0, \infty). \end{cases}$$

We define functions which play a crucial role in the asymptotic analysis (cf. [**1**]):

$$L(x) := \lambda'(x)\left(\frac{g'(x)}{\lambda'(x)}\right)', \qquad G(x) := \left(\frac{\lambda(x)}{\lambda'(x)}\right)^2 L(x),$$
$$l(x) := \frac{1}{\sqrt{L(x)}}.$$

Suppose in addition that

$$a_n \sim e^{-g(n)} \quad \text{as } n \to \infty, \quad \text{and} \quad \lambda_n = \lambda(n) \quad \text{for } n \geq x_0 \in \mathbf{N}.$$

Our primary purpose is to prove three theorems concerning *one-sided* Tauberian conditions on $\{s_n\}$ under which $s_n \to s\,(D_{\lambda,a})$ implies $s_n \to s$. These theorems generalize two-sided or O-Tauberian results proved in [**1**].

By [**1**, Lemma 3] we have that $\sigma := -\lim_{x\to\infty}(g'(x)/\lambda'(x))$ is the abscissa of convergence of $a(x)$ and that $\lim_{x\to\sigma+} a(x) = \infty$. (As noted in the Remark after Lemma 3 in [**1**] the proof of that lemma does not require L or G to be monotonic. There is a misprint in the proof of that lemma. On page 161, line 3 of [**1**] it should be $e^{\beta h_2(k,x)}$ instead of $e^{-\beta h_2(k,x)}$).

The following three theorems are our main results:

Theorem 1. *Assume* (C), *and suppose that*

$$\frac{\lambda'(x)}{\lambda(x)} \to 0 \quad \textit{and} \quad G(x) \to \delta \in (0,\infty) \quad \textit{as } x \to \infty,$$

and that $s_n \to s\,(D_{\lambda,a})$ *and*

$$\lim_{\varepsilon\to 0+} \liminf_{n\to\infty} \min_{n\le m\le n+\varepsilon l(n)} (s_{m+1} - s_n) \ge 0. \tag{1}$$

Then $s_n \to s$.

Theorem 2. *Assume* (C), *and suppose that* $L(x)$ *is nonincreasing and* $G(x)$ *is nondecreasing on* $[x_0,\infty)$ *with* $L(x) \to 0$ *and* $G(x) \to \infty$ *as* $x \to \infty$, *and that* $s_n \to s\,(D_{\lambda,a})$ *and* (1) *holds. Then* $s_n \to s$.

Theorem 3. *Let* $A_n := \min(e^{\alpha_n}, e^{\beta_n})$ *for* $n > x_0$, *where*

$$\alpha_n := g(n) - g(n-1) - \big(\lambda(n) - \lambda(n-1)\big)\,\frac{g'(n-1)}{\lambda'(n-1)},$$

and

$$\beta_n := g(n-1) - g(n) + \big(\lambda(n) - \lambda(n-1)\big)\,\frac{g'(n)}{\lambda'(n)}.$$

Assume (C), *and suppose that* $L(x) \ge \delta > 0$ *on* $[x_0,\infty)$, *and that* $s_n \to s\,(D_{\lambda,a})$ *and*

$$s_n - s_{n-1} \ge -cA_n \quad \textit{for } n > x_0, \tag{2}$$

where c *is a positive constant. Then* $s_n \to s$.

Remarks. Theorem 2 with the more restrictive O-Tauberian condition

$$\lim_{\varepsilon\to 0+} \limsup_{n\to\infty} \max_{n\le m\le n+\varepsilon l(n)} |s_{m+1} - s_n| = 0 \tag{1'}$$

in place of (1) is proved as Theorem 1 in [**1**]; and Theorem 3 with the O-Tauberian condition

$$s_n - s_{n-1} = O(A_n), \tag{2'}$$

in place of (2) is proved as Theorem 2 in [**1**]. There is no counterpart to Theorem 1 in [**1**].

The three theorems deal with rates of increase, respectively decrease, of the sequence of weights $\{a_n\}$ with respect to the "gap-sequence" $\{\lambda_n\}$. Though the same Tauberian condition is used in Theorems 1 and 2, the methods of proof are different in the two cases. Theorem 1 deals essentially with the situation where $\sigma = 0$, and the function $S(t) := \sum_{\lambda_k \le t} a_k$ is regularly varying with index larger than 1 (for the notation see, e.g., [**2**]); whereas Theorem 2 handles the cases where $S(t)$ increases more rapidly, or $S(\infty) - S(t)$ decreases more rapidly than any power of t, but where we are not yet in the so-called high index case which is finally considered in Theorem 3.

In order to get some insight into the results and to compare them with results in the literature we shall now discuss some special gap-sequences $\{\lambda_n\}$.

(a) $\lambda_n = n$. In this case we have $L(x) = g''(x)$, $l(x) = 1/\sqrt{g''(x)}$, $G(x) = x^2 g''(x)$, and the Dirichlet series summability method reduces to the power series method, that is, $a(x) = \sum_{k=1}^{\infty} a_k t^k$ with $t = e^{-x}$. Without loss of generality we may assume the radius of convergence of the power series is either $R = 1$ (i.e., $-\sigma = \lim_{x\to\infty} g'(x) = 0$) or $R = \infty$ (i.e., $-\sigma = \lim_{x\to\infty} g'(x) = \infty$). When $R = 1$ we get Abel-type summability methods, and when $R = \infty$ we get Borel-type methods. Tauberian theorems for these methods, in particular the Abel and Borel methods, have a long history beginning a century ago with Tauber's result on the Abel method followed by Hardy's and Littlewood's results on that method. Oscillation conditions as used in Theorems 1 and 2 were introduced by Landau [**20**] and Schmidt [**21**], [**22**]. General power series methods with regularly varying weights $\{a_n\}$ were studied by Jakimovski, Tietz and Trautner [**11**], [**25**], and our Theorem 1 under slightly different assumptions applies to their results. More general classes of weights are discussed in Kales [**13**] and in [**16**]–[**18**]. The latter results use two-sided conditions, while in [**14**], [**15**] the corresponding one-sided results are proved. Our Theorems 1 and 2 cover these results. Actually, also for power series methods the case of regularly varying weights needs a different treatment from that for other weights. A high-indices theorem for power series methods is given in [**5**]; however, only two-sided conditions are used, i.e., $s_n - s_{n-1} = O(A_n)$, while our Theorem 3 deals with the corresponding one-sided Tauberian condition for this case.

Some results on Dirichlet methods can be found, e.g., in [**3**], [**4**], [**23**].

(b) $\lambda_n = n^\alpha$ with $\alpha > 0$. As above, we consider the cases $\sigma = 0$ and $\sigma = -\infty$ separately.

(b1) $\sigma = 0$. In this case we have $g'(x)x^{1-\alpha} \to 0$ as $x \to \infty$, e.g., $g'(x) = -x^{\alpha-\beta-1}$ with $\beta > 0$. If $\beta = \alpha$, then $L(x) = \alpha x^{-2}$ and Theorem 1 applies with $l(n) \asymp n$. If $\beta < \alpha < \beta + 2$, then Theorem 2 applies with $l(n) \asymp n^{1-(\alpha-\beta)/2}$, and finally if $\alpha \geq \beta+2$, then Theorem 3 applies with $\log A_n \sim (\beta/2)n^{\alpha-\beta-2}$.

(b2) $\sigma = -\infty$. Here we have $g'(x)x^{1-\alpha} \to \infty$ as $x \to \infty$, e.g., $g'(x) = x^{\alpha+\beta-1}$ with $\beta > 0$. Now Theorem 1 does not apply. If $\alpha + \beta < 2$, then Theorem 2 applies with $l(n) \asymp n^{1-(\alpha+\beta)/2}$, and if $\alpha + \beta \geq 2$, then Theorem 3 applies with $\log A_n \sim (\beta/2)n^{\alpha+\beta-2}$.

(c) $\lambda_n = e^{\sqrt{n}}$. Again we consider the cases $\sigma = 0$ and $\sigma = -\infty$ separately.

(c1) $\sigma = 0$. In this case we have $g'(x)\sqrt{x}e^{-\sqrt{x}} \to 0$ as $x \to \infty$, e.g., $g'(x) = -x^\alpha$, so that $L(x) \sim (1/2)x^{\alpha-(1/2)}$ as $x \to \infty$. If $\alpha = -1/2$, then Theorem 1 applies with $l(n) \asymp \sqrt{n}$. If $-1/2 < \alpha < 1/2$, then Theorem 2 applies with $l(n) \asymp n^{(1/4)-(1/2)\alpha}$, and finally, if $\alpha \geq 1/2$, then Theorem 3 applies with $\log A_n \sim (1/4)n^{\alpha-(1/2)}$.

(c2) $\sigma = -\infty$. Here we have $g'(x)\sqrt{x}e^{-\sqrt{x}} \to \infty$ as $x \to \infty$, e.g., $g'(x) = x^{\alpha-(1/2)}\, e^{\sqrt{x}}$ with $\alpha > 1/2$, so that $L(x) \sim \alpha x^{\alpha-(3/2)}e^{\sqrt{x}}$ as $x \to \infty$, and Theorem 3 applies with $\log A_n \sim (\alpha/2)n^{\alpha-(3/2)}e^{\sqrt{n}}$.

(d) $\lambda_n = e^n$. Now $1 = (\lambda'(x)/\lambda(x)) \not\to 0$, $G(x) = L(x)$, so that neither Theorem 1 nor Theorem 2 is applicable. For $g(x) = -\alpha x$ with $\alpha > 0$, we have $L(x) = \alpha$, and we can apply Theorem 3 with $A_n \asymp 1$.

Note that we always need growth conditions on $s_n - s_{n-1}$, so we do not get a high indices theorem without such conditions, in contrast to what has been shown for the Abel method by Hardy and Littlewood [**9**], for the Borel method by Gaier [**7**], for the logarithmic method by Krishnan [**19**], and for a somewhat larger class of methods by Jakimovski, Meyer-

König and Zeller [**12**].

(e) $\lambda_n = \log(n+1)$. This gap-sequence is not in the range of our theorems because $x(\lambda'(x)/\lambda(x)) \searrow 0$ as $x \to \infty$, and this violates one of the conditions in (C).

2. Proofs of the main results. Our main tool in the proof of Theorem 1 is the following result due to Borwein [**4**, Theorem 6]:

Lemma 1. *Suppose that the abscissa* $\sigma = 0$, *that*

$$A_n := \sum_{k=1}^{n} a_k \to \infty, \quad \lambda_{n+1} \sim \lambda_n,$$

$$\frac{A_m}{A_n} \to 1 \quad \textit{when}\ \frac{\lambda_m}{\lambda_n} \to 1,\ m > n \to \infty,$$

$$\liminf(s_m - s_n) \geq 0 \quad \textit{when}\ \frac{A_m}{A_n} \to 1,\ m > n \to \infty,$$

and that $s_n \to s\,(D_{\lambda,a})$. *Then* $s_n \to s$.

Proof of Theorem 1. Assume the hypotheses of Theorem 1. Then, for $x \geq x_0$,

$$\frac{g'(x)}{\lambda'(x)} - \frac{g'(x_0)}{\lambda'(x_0)} = \int_{x_0}^{x} G(t)\frac{\lambda'(t)}{\lambda^2(t)}\,dt \leq \sup_{x \geq x_0} G(x) \int_{x_0}^{\infty} \frac{\lambda'(t)}{\lambda^2(t)}\,dt < \infty,$$

so that $\sigma = -\lim_{x\to\infty}(g'(x)/\lambda'(x))$ is finite. Moreover, for $x \geq x_0$,

$$-\frac{g'(x)}{\lambda'(x)} - \sigma = \int_{x}^{\infty} G(t)\frac{\lambda'(t)}{\lambda^2(t)}\,dt := \frac{\delta_1(x)}{\lambda(x)},$$

where $\delta_1(x) \to \delta$ as $x \to \infty$. Hence, for $x \geq x_0$,

$$\begin{aligned}-g(x) + g(x_0) &= -\int_{x_0}^{x} \lambda'(t)\frac{g'(t)}{\lambda'(t)}\,dt = \int_{x_0}^{x} \left(\sigma\lambda'(t) + \delta_1(t)\frac{\lambda'(t)}{\lambda(t)}\right) dt \\ &=: \sigma\lambda(x) + \delta(x)\log\lambda(x),\end{aligned}$$

where $\delta(x) \to \delta$ as $x \to \infty$. Consequently,

$$a_k \sim e^{-g(k)} = e^{-g(x_0)} e^{\sigma\lambda(k)} (\lambda(k))^{\delta(k)} \quad \text{as } k \to \infty,$$

and

$$a(x) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k x} \sim e^{-g(x_0)} \sum_{k=x_0}^{\infty} e^{\lambda(k)(\sigma - x)} (\lambda(k))^{\delta(k)} \quad \text{as } x \to \sigma + .$$

Thus we may assume, without loss of generality, that $\sigma = 0$ and $g(x_0) = 0$, so that

$$a_k \sim (\lambda(k))^{\delta(k)} \quad \text{as } k \to \infty,$$

and

$$-g(x) = \delta(x) \log \lambda(x) \in C_2[x_0, \infty),$$

where $\delta(x) \to \delta$ as $x \to \infty$. Consequently $a_k = f(\lambda(k))$ with some regularly varying function f. We then have:

(i)

$$\frac{\lambda_{n+1}}{\lambda_n} = \exp\left\{ \int_n^{n+1} \frac{\lambda'(t)}{\lambda(t)}\, dt \right\} \to 1 \quad \text{as } n \to \infty;$$

(ii)

$$A_n := \sum_{k=1}^{n} a_k \sim \sum_{k=x_0}^{n} (\lambda(k))^{\delta(k)} \to \infty \quad \text{as } n \to \infty.$$

Suppose now that

(3) $$m > n \to \infty \quad \text{with } \frac{\lambda_m}{\lambda_n} \to 1.$$

Then

$$(m-n)\frac{\lambda'(m)}{\lambda(m)} \le \int_n^m \frac{\lambda'(t)}{\lambda(t)}\, dt = \log \frac{\lambda_m}{\lambda_n} \to 0.$$

Hence

$$0 < \varepsilon(m,n) := \frac{m-n}{l(n)} = (m-n)\frac{\lambda'(n)}{\lambda(n)} \cdot \frac{\lambda(n)}{l(n)\lambda'(n)} \to 0,$$

since

$$l(n)\frac{\lambda'(n)}{\lambda(n)} = \frac{1}{\sqrt{G(n)}} \to \frac{1}{\sqrt{\delta}}.$$

Thus, subject to (3),

$$m = n + \varepsilon(m,n)l(n) \quad \text{with } \varepsilon(m,n) \to 0.$$

Further

$$0 \le A_m - A_n = \sum_{k=n+1}^{m} a_k \sim \sum_{k=n+1}^{m} e^{-g(k)} \sim \int_n^m e^{-g(t)}\,dt,$$

because

$$-g'(t) = \lambda'(t)\int_t^\infty G(u)\frac{\lambda'(u)}{\lambda^2(u)}\,du = \delta_1(t)\frac{\lambda'(t)}{\lambda(t)} > 0 \quad \text{on } [x_0,\infty)$$

with $\delta_1(t) \to \delta$ as $t \to \infty$, so that

$$\begin{aligned}\frac{e^{-g(k+1)}}{e^{-g(k)}} &= \exp\left(-\int_k^{k+1} g'(t)\,dt\right)\\ &= \exp\left(\int_k^{k+1} \delta_1(t)\frac{\lambda'(t)}{\lambda(t)}\,dt\right) \to 1 \quad \text{as } k \to \infty.\end{aligned}$$

Also,

$$\int_n^m e^{-g(t)}\,dt \sim (m-n)e^{-g(n)} \sim \varepsilon(m,n)l(n)a_n,$$

since

$$\begin{aligned}\frac{e^{-g(m)}}{e^{-g(n)}} &= \exp\left(-\int_n^m g'(t)\,dt\right)\\ &= \exp\left(\int_n^m \delta_1(t)\frac{\lambda'(t)}{\lambda(t)}\,dt\right) \sim \left(\frac{\lambda_m}{\lambda_n}\right)^\delta \to 1.\end{aligned}$$

Next, since

$$\frac{l(n)}{n} \sim \frac{1}{\sqrt{\delta}}\frac{\lambda(n)}{n\lambda'(n)} \le \frac{1}{\sqrt{\delta}}\frac{\lambda(x_0)}{x_0\lambda'(x_0)},$$

we have that, for a sufficiently small positive constant c,

$$A_n \sim \sum_{k=x_0}^{n} e^{-g(k)} \ge \int_{n-cl(n)}^{n} e^{-g(t)}\,dt \sim cl(n)e^{-g(n)},$$

and so, subject to (3),

$$0 \le A_m - A_n \sim \varepsilon(m,n)l(n)e^{-g(n)} = o(A_n), \quad \text{whence } \frac{A_m}{A_n} \to 1.$$

We have thus shown:

(iii) If $m > n \to \infty$ with $(\lambda_m/\lambda_n) \to 1$, then $m = n + \varepsilon(m,n)l(n)$ where $\varepsilon(m,n) \to 0$, and $(A_m/A_n) \to 1$.

Observe now that:

(iv) If $m > n \to \infty$ with $(A_m/A_n) \to 1$, then

$$A_n \sim -\int_{x_0}^{n} e^{-g(t)} \frac{g'(t)\lambda(t)}{\delta\lambda'(t)}\,dt \le -\frac{\lambda(n)}{\delta\lambda'(n)} \int_{x_0}^{n} e^{-g(t)} g'(t)\,dt$$
$$\sim \frac{1}{\sqrt{\delta}}\, l(n)e^{-g(n)},$$

and

$$A_m - A_n \sim \int_n^m e^{-g(t)}\,dt \ge (m-n)e^{-g(n)},$$

so that, for a sufficiently large positive constant c,

$$0 < \varepsilon(m,n) := \frac{m-n}{l(n)} = \frac{m-n}{A_n}\, e^{-g(n)} \cdot \frac{A_n}{l(n)}\, e^{g(n)} \le c\,\frac{A_m - A_n}{A_n} \to 0,$$

i.e., $m = n + \varepsilon(m,n)l(n)$, where $\varepsilon(m,n) \to 0$.

It follows from (iv) and condition (1) that:

(v) $\liminf(s_m - s_n) \ge 0$ when $m > n \to \infty$ with $(A_m/A_n) \to 1$. By virtue of (i), (ii), (iii) and (v) we have, by Lemma 1, that $s_n \to s\ (D_{\lambda,a})$ implies $s_n \to s$. □

Proof of Theorem 2. Suppose that $s_n \to s\ (D_{\lambda,a})$ and that (1) holds. Then, by Theorem 4 below, $s_n = O(1)$. Therefore, by [**1**, Proposition]

$$\lim_{n\to\infty} f_n(\alpha) = s \quad \text{for all } \alpha > 0, \tag{4}$$

where

$$f_n(\alpha) := \frac{1}{\tilde{a}_\alpha(\tau_n(\alpha))} \sum_{k=x_0}^{\infty} s_k e^{-\alpha g(k)}\, e^{-\lambda(k)\tau_n(\alpha)},$$

$$\tilde{a}_\alpha(x) := \sum_{k=x_0}^{\infty} e^{-\alpha g(k)}\, e^{-\lambda(k)x}$$

and

$$\tau_n(\alpha) := -\,\alpha\, \frac{g'(n)}{\lambda'(n)}.$$

Next, our assumptions imply that, as $x \to \infty$,

$$(5) \qquad l(x) = \frac{1}{\sqrt{L(x)}} \nearrow \infty \quad \text{and} \quad \frac{l(x)}{x} = \frac{\lambda(x)}{x\lambda'(x)} \frac{1}{\sqrt{G(x)}} \searrow 0.$$

We may assume that $s = 0$, so that we have to show that $s_n \to 0$. We proceed as in [**15**, Section 3], [**14**, Section 3.4]. Suppose, in contradiction to what we wish to prove, that $s_{m_k} \geq \zeta$ for some $\zeta > 0$ and a sequence of integers $\{m_k\}$ with $1 \leq m_1 < m_2 < \cdots$. By (1) there exists $\varepsilon > 0$ such that

$$\liminf_{n\to\infty} \min_{n \leq m \leq n+4\varepsilon l(n)} (s_{m+1} - s_n) \geq -\frac{1}{3}\zeta,$$

so that, for sufficiently large N,

$$s_{m+1} - s_n \geq -\frac{1}{2}\zeta$$

whenever $N \leq n \leq m \leq n + 4\varepsilon l(n)$. Hence

$$s_{m+1} = s_{m+1} - s_{m_k} + s_{m_k} \geq \frac{1}{2}\zeta$$
$$\text{for } N \leq m_k \leq m \leq m_k + 4\varepsilon\, l(m_k).$$

Define $n_k := m_k + [2\varepsilon l(m_k)]$. If $n_k - \varepsilon l(n_k) \leq m \leq n_k + \varepsilon l(n_k)$ and if k is sufficiently large, then it follows (in view of (5) and since $1 \leq (l(n_k)/l(m_k)) \leq (n_k/m_k) \to 1$) that $m \leq m_k + 4\varepsilon l(m_k)$, and that $m - 1 \geq m_k$ so that $s_m \geq \zeta/2$. Since $s_n = O(1)$, it follows from [**1**, Theorem A and Lemmas 4, 5, 7, 9 and 11] that for all $\alpha > 0$,

$$\lim_{n\to\infty} \left| f_n(\alpha) - \sqrt{\frac{\alpha}{2\pi}}\, \frac{1}{l(n)} \int_{n-\delta(n)}^{n+\delta(n)} e^{-\alpha L(n)(t-n)^2/2} s(t)\, dt \right| = 0,$$

where $s(t) := s_k$ for $k \leq t < k+1$, $\delta(x) := (\gamma/10)(\lambda(x)/\lambda'(x))$, and $\gamma := \min(1, (x_0\lambda'(x_0)/\lambda(x_0)))$. Substituting $v = (t-n)\sqrt{\alpha L(n)}$, we get that

$$\int_{n-\delta(n)}^{n-\varepsilon l(n)} e^{-\alpha L(n)(t-n)^2/2}\,dt \leq \frac{l(n)}{\sqrt{\alpha}}\int_{-\infty}^{-\varepsilon\sqrt{\alpha}} e^{-v^2/2}\,dv.$$

Hence, for some constant $c > 0$ which does not depend on α and $n = n_k$,

$$\begin{aligned}
&\limsup_{n\to\infty} f_n(\alpha)\\
&\geq \limsup_{n\to\infty}\sqrt{\frac{\alpha}{2\pi}}\,\frac{1}{l(n)}\int_{n-\varepsilon l(n)}^{n+\varepsilon l(n)} e^{-(1/2)\alpha L(n)(t-n)^2} s(t)\,dt - c\int_{\varepsilon\sqrt{\alpha}}^{\infty} e^{-(1/2)v^2}\,dv\\
&\geq \limsup_{n\to\infty}\frac{\zeta}{2}\sqrt{\frac{\alpha}{2\pi}}\,\frac{1}{l(n)}\int_{n-\varepsilon l(n)}^{n+\varepsilon l(n)} e^{-(1/2)\alpha L(n)(t-n)^2}\,dt - c\int_{\varepsilon\sqrt{\alpha}}^{\infty} e^{-(1/2)v^2}\,dv\\
&= \frac{\zeta}{2}\,\frac{1}{\sqrt{2\pi}}\int_{-\varepsilon\sqrt{\alpha}}^{\varepsilon\sqrt{\alpha}} e^{-(1/2)v^2}\,dv - c\int_{\varepsilon\sqrt{\alpha}}^{\infty} e^{-(1/2)v^2}\,dv \to \frac{\zeta}{2} > 0 \quad \text{as } \alpha\to\infty,
\end{aligned}$$

which contradicts (4) with $s = 0$. This establishes the desired result. □

Proof of Theorem 3. Suppose that $s_n \to s\,(D_{\lambda,a})$ and that (2) holds. Then, by Theorem 5 below, $s_n = O(1)$. Since $A_n > 1$ for all $n > x_0$, it follows that $s_n - s_{n-1} = O(A_n)$. Hence, by [**1**, Theorem 2], $s_n \to s$, and this completes the proof. □

3. Vijayaraghavan-type results. In this section we prove two theorems. The proof of the first of these uses Vijayaraghavan's theorem [**8**], [**26**], [**27**] directly, and the proof of the second is based on the method of proof of Vijayaraghavan's theorem in [**8**].

Theorem 4. *Assume* (C), *and suppose that* $L(x)$ *is nonincreasing and* $G(x)$ *is nondecreasing on* $[x_0,\infty)$ *with* $L(x)\to 0$ *and* $G(x)\to\infty$ *as* $x\to\infty$, *that* $s_n = O(1)\,(D_{\lambda,a})$, *and that*

$$(1*)\qquad \liminf_{n\to\infty}\min_{n\leq m\leq n+l(n)}(s_{m+1}-s_n) > -\infty.$$

Then $s_n = O(1)$.

The main tool required for the proof of Theorem 4 is the following variant of a result originally given by Vijayaraghavan [**26**], [**27**]. It is stated in [**14**] and [**15**] and can be established by slightly modifying the proof of Theorem 238 in [**8**].

Lemma 2. *Let* $s(t) := s_n$ *for* $n \le t < n+1$, $n = 1, 2, \ldots$, *and suppose that* $c_n(t)$ *and* $\phi(t)$ *are functions on* $[t_0, \infty)$ *for some* $t_0 > 0$ *satisfying*:

(i) *for* $n = 1, 2, \ldots$, $c_n(t) \ge 0$ *on* $[t_0, \infty)$, $c_n(t) \to 0$, $\sum_{k=1}^{\infty} c_k(t) \to 1$ *as* $t \to \infty$;

(ii) $\phi(t)$ *is positive, strictly increasing and unbounded and* $\phi(t+1) - \phi(t) \le 2$ *on* $[t_0, \infty)$;

(iii) $\sum_{k=1}^{M} c_k(t) \to 0$ *when* $\phi(t) - \phi(M) \to \infty$, $t > M \to \infty$;

(iv) $\sum_{k=N}^{\infty} c_k(t)\{\phi(k) - \phi(N)\} \to 0$ *when* $\phi(N) - \phi(t) \to \infty$, $N > t \to \infty$;

(v) $s(u) - s(t) > -a\{\phi(u) - \phi(t)\} - b$ *for* $u \ge t > t_0$, *where* a, b *are positive constants.*

Suppose also that $\sum_{k=1}^{\infty} c_k(t) s_k$ *is convergent and its sum is bounded for* $t \ge t_0$. *Then* $s_n = O(1)$.

We also need the following result:

Lemma 3. *Under the assumptions of Theorem* 2 *or Theorem* 4,

$$\Delta_n := \inf_{x > \sigma} a(x) e^{\lambda_n x} \sim \sqrt{2\pi}\, a_n l(n),$$

and

$$\sum_{k=1}^{\infty} \frac{a_k}{\Delta_k} = \infty.$$

Moreover, for large n,

$$\Delta_n = a(x_n) e^{\lambda_n x_n}, \quad x_n = -\frac{g'(t_n)}{\lambda'(t_n)},$$

where the sequence $\{x_n\}$ *is ultimately nonincreasing with* $x_n \to \sigma$,

$$\sigma = \lim_{t\to\infty}\left(-\frac{g'(t)}{\lambda'(t)}\right),$$

and $\{t_n\}$ *is ultimately nondecreasing with* $\sqrt{L(n)}\,|n-t_n| \to 0$.

Proof. First, choose n_0 such that $a_n > 0$ for all $n \geq n_0 - 1$. Then, for $n \geq n_0$, we have that $a(x)e^{\lambda_n x} \geq a_{n-1}e^{(\lambda_n-\lambda_{n-1})x} \to \infty$ as $x \to \infty$. Also, if $\sigma > -\infty$, then $a(x)e^{\lambda_n x} \geq a(x)e^{\lambda_n \sigma} \to \infty$ as $x \to \sigma+$, since $a(x) \to \infty$ as $x \to \sigma+$; and if $\sigma = -\infty$, then $a(x)e^{\lambda_n x} \geq a_{n+1}e^{(\lambda_n-\lambda_{n+1})x} \to \infty$ as $x \to \sigma+$. Hence, for every $n \geq n_0$, there exists a finite $x_n > \sigma$ such that $\Delta_n = a(x_n)e^{\lambda_n x_n}$. Next, for $m, n \geq n_0$, we have that $\Delta_m \leq a(x_n)e^{\lambda_m x_n} = \Delta_n e^{(\lambda_m-\lambda_n)x_n}$, whence, by symmetry,

$$e^{(\lambda_m-\lambda_n)x_m} \leq \frac{\Delta_m}{\Delta_n} \leq e^{(\lambda_m-\lambda_n)x_n}.$$

It follows that, if $n > m \geq n_0$, then $x_m > x_n$, i.e., the sequence $\{x_n\}$ is ultimately nonincreasing and so tends to a limit $\rho \geq \sigma$. Assume, if possible, that $\rho > \sigma$. Then $x_{n_0} \geq x_n \geq \rho$ for all $n \geq n_0$, and $\rho - \varepsilon > \sigma$ for some $\varepsilon > 0$. Hence, by the definition of x_n,

$$1 \geq \frac{a(x_n)e^{\lambda_n x_n}}{a(\rho-\varepsilon)e^{\lambda_n(\rho-\varepsilon)}} \geq \frac{a(x_n)}{a(\rho-\varepsilon)}e^{\varepsilon\lambda_n} \sim \frac{a(\rho)}{a(\rho-\varepsilon)}e^{\varepsilon\lambda_n} \to \infty$$
$$\text{as } n \to \infty,$$

which is a contradiction. Thus $x_n \to \rho = \sigma$ as $n \to \infty$ (cf., [**16**]–[**18**]). Observe also that, by [**1**, Lemma 3], $\sigma = \lim_{t\to\infty}(-(g'(t)/\lambda'(t)))$.

Assume throughout the rest of the proof that n is large. By (C), we now get that $x_n = -(g'(t_n)/\lambda'(t_n))$, where $\{t_n\}$ is nondecreasing and unbounded. Let $\tilde{a}(x) := \sum_{k=x_0}^{\infty} e^{-g(k)}e^{-\lambda(k)x}$. Since $a_n \sim e^{-g(n)}$ and $t_n \nearrow \infty$, it follows from the regularity of the $D_{\lambda,a}$ method and [**1**, Theorem A] that

(6)

$$\begin{cases} \dfrac{\Delta_n}{a_n} = \dfrac{a(x_n)}{a_n}\,e^{\lambda_n x_n} \sim \tilde{a}\left(-\dfrac{g'(t_n)}{\lambda'(t_n)}\right)\exp\left(g(n) - \lambda(n)\,\dfrac{g'(t_n)}{\lambda'(t_n)}\right) \\ \qquad \sim \sqrt{2\pi}\; l(t_n)e^{-h_1(t_n,n)} \end{cases}$$

where, for all $t, x \geq x_0$,

$$h_1(t,x) := g(t) - g(x) + \big(\lambda(x) - \lambda(t)\big)\frac{g'(t)}{\lambda'(t)}$$
$$= -\int_t^x \int_u^x \frac{\lambda'(v)}{\lambda'(u)} L(u)\,dv\,du \leq 0.$$

Let $\tilde{\delta} = \tilde{\delta}(x) := \tilde{\gamma}(\lambda(x)/\lambda'(x))$ with $0 < \tilde{\gamma} \leq (\gamma/10) = (1/10)$ $\min(1, x_0(\lambda(x_0)/\lambda'(x_0)))$, and let $x_0 \leq t \leq x - \tilde{\delta}$, $x_0 \leq u \leq x - (\tilde{\delta}/4)$. Then, by [**1**, (17) and (19)],

$$h_1(t,x) \leq h_1(x - \tilde{\delta}, x) \leq -\frac{\tilde{\delta}^2}{4} L(x) = -\frac{\tilde{\gamma}^2}{4} G(x),$$

$$\frac{\lambda(x)}{\lambda(u)} \geq \exp\left(\int_{x-(\tilde{\delta}/4)}^x \frac{\lambda'(v)}{\lambda(v)}\,dv\right) \geq \exp\left(\frac{\tilde{\gamma}}{4}\frac{\lambda(x)}{\lambda'(x)}\frac{\lambda'(x)}{\lambda(x)}\right) \geq 1 + \frac{\tilde{\gamma}}{4},$$

$$\log\frac{\lambda(x)}{\lambda(x - (\tilde{\delta}/4))} = \int_{x-(\tilde{\delta}/4)}^x \frac{\lambda'(v)}{\lambda(v)}\,dv \leq \frac{x\lambda'(x)}{\lambda(x)}\,\frac{(\tilde{\delta}/4)}{x - (\tilde{\delta}/4)}$$
$$\leq \frac{\tilde{\delta}}{2}\frac{\lambda'(x)}{\lambda(x)} \leq \frac{1}{2}\int_{x-\tilde{\delta}}^x \frac{\lambda'(v)}{\lambda(v)}\,dv = \frac{1}{2}\log\frac{\lambda(x)}{\lambda(x-\tilde{\delta})}.$$

Hence

$$-h_1(t,x) = \int_t^x L(v)\frac{\lambda(x) - \lambda(v)}{\lambda'(v)}\,dv$$
$$\geq \frac{\tilde{\gamma}}{4}\int_t^{x-(\tilde{\delta}/4)} L(v)\frac{\lambda(v)}{\lambda'(v)}\,dv \geq \frac{\tilde{\gamma}}{4} G(t)\int_t^{x-(\tilde{\delta}/4)} \frac{\lambda'(v)}{\lambda(v)}\,dv$$
$$= \frac{\tilde{\gamma}}{4} G(t)\left(\log\frac{\lambda(x)}{\lambda(t)} - \log\frac{\lambda(x)}{\lambda(x - (\tilde{\delta}/4))}\right)$$
$$\geq \frac{\tilde{\gamma}}{4} G(t)\left(\log\frac{\lambda(x)}{\lambda(t)} - \frac{1}{2}\log\frac{\lambda(x)}{\lambda(x-\tilde{\delta})}\right) \geq \frac{\tilde{\gamma}}{8} G(t)\log\frac{\lambda(x)}{\lambda(t)}.$$

In addition,

$$\log\frac{\lambda(x)}{\lambda(t)} = \int_t^x \frac{1}{v}\frac{v\lambda'(v)}{\lambda(v)}\,dv \geq \gamma\int_t^x \frac{1}{v}\,dv = \gamma\log\frac{x}{t}.$$

Because $G(t) \to \infty$ as $t \to \infty$, we now obtain, for sufficiently large t and $x \geq t + \tilde{\delta}$, that

$$\begin{aligned}\frac{l(t)}{l(x)}\, e^{-h_1(t,x)} &= \exp\Bigg(- h_1(t,x) + \log \frac{x\,\lambda'(x)\,\lambda(t)}{t\,\lambda'(t)\,\lambda(x)} \\ &\qquad\qquad - \log \frac{x}{t} + \frac{1}{2} \log \frac{G(x)}{G(t)}\Bigg) \\ &\geq \exp\left(\frac{\tilde{\gamma}^2}{8}\, G(x) + \frac{\tilde{\gamma}}{16}\, G(t) \log \frac{\lambda(x)}{\lambda(t)} - \log \frac{x}{t}\right) \\ &\geq \exp\left(\frac{\tilde{\gamma}^2}{8}\, G(x)\right) \to \infty \quad \text{as } x \to \infty.\end{aligned}$$

It follows that $(l(\tau_n)/l(n))e^{-h_1(\tau_n,n)} \to \infty$ for any sequence $\{\tau_n\}$ with $\tau_n \to \infty$ and $\tau_n \leq n - \tilde{\gamma}(\lambda(n)/\lambda'(n))$. Moreover, if $|t - x| \geq cl(x)$ for some $c > 0$, then, as above, by [**1**, (17) and (19)],

$$- h_1(t,x) \geq \frac{1}{4}\, L(x)(cl(x))^2 = \frac{c^2}{4} > 0.$$

Suppose now that

$$\tau_n \to \infty \quad \text{and} \quad \limsup_{n\to\infty} |\tau_n - n|\sqrt{L(n)} > 0,$$

so that $|\tau_n - n| \geq cl(n)$ for some $c > 0$ and infinitely many n, and hence,

$$- h_1(\tau_n, n) \geq \frac{c^2}{4} \quad \text{for all such } n.$$

We consider three cases for these n.

(a) $\tau_n \geq n$ infinitely often. Then

$$\limsup_{n\to\infty} \frac{l(\tau_n)}{l(n)}\, e^{-h_1(\tau_n,n)} \geq \limsup_{n\to\infty} e^{-h_1(\tau_n,n)} \geq e^{c^2/4} > 1.$$

(b) $\tau_n \leq n - \tilde{\gamma}(\lambda(n)/\lambda'(n))$ infinitely often for some $\tilde{\gamma} > 0$. In this case we have, as shown above, that

$$\limsup_{n\to\infty} \frac{l(\tau_n)}{l(n)}\, e^{-h_1(\tau_n,n)} = \infty.$$

(c) $n-\varepsilon(\lambda(n)/\lambda'(n)) \le \tau_n \le n$ infinitely often for an arbitrary $\varepsilon > 0$. Then

$$l(n) \ge l(\tau_n) = \frac{\lambda(\tau_n)\tau_n}{\sqrt{G(\tau_n)}\,\lambda'(\tau_n)\tau_n} \ge \frac{\lambda(n)}{\sqrt{G(n)}\,\lambda'(n)n}\tau_n$$
$$\ge l(n)\,\frac{n-\varepsilon\lambda(n)/\lambda'(n)}{n} \ge l(n)\left(1-\frac{\lambda(x_0)}{x_0\lambda'(x_0)}\,\varepsilon\right).$$

Hence,

$$\limsup_{n\to\infty}\frac{l(\tau_n)}{l(n)}\,e^{-h_1(\tau_n,n)} \ge \left(1-\frac{\lambda(x_0)}{x_0\lambda'(x_0)}\,\varepsilon\right)\limsup_{n\to\infty} e^{-h_1(\tau_n,n)}$$
$$\ge \left(1-\frac{\lambda(x_0)}{x_0\lambda'(x_0)}\,\varepsilon\right)e^{(1/4)c^2} > 1,$$

provided ε is sufficiently small. We have thus shown that

$$\limsup_{n\to\infty}\frac{l(\tau_n)}{l(n)}\,e^{-h_1(\tau_n,n)} > 1,$$

if $\tau_n \to \infty$ and $\limsup_{n\to\infty} |\tau_n - n|\sqrt{L(n)} > 0$.

Finally, let $\tilde{x}_n := -(g'(n)/\lambda'(n))$. Then, by [**1**, Theorem A], the definition of Δ_n, and (6), we have that

$$\sqrt{2\pi}l(n) \sim \frac{a(\tilde{x}_n)}{a_n}\,e^{\lambda_n\tilde{x}_n} \ge \frac{a(x_n)}{a_n}\,e^{\lambda_n x_n} = \frac{\Delta_n}{a_n}$$
$$\sim \sqrt{2\pi}\,l(t_n)e^{-h_1(t_n,n)}.$$

Hence

$$\limsup_{n\to\infty}\frac{l(t_n)}{l(n)}\,e^{-h_1(t_n,n)} \le 1.$$

Since $t_n \to \infty$ it follows that we must have

$$\lim_{n\to\infty}|t_n - n|\sqrt{L(n)} = 0.$$

Next, we have, by [**1**, (17) and (19)] that, for some ξ, ζ lying between t_n and n,

$$|h_1(t_n,n)| = \frac{1}{2}\,\frac{\lambda'(\xi)}{\lambda'(\zeta)}\,L(\zeta)(t_n-n)^2 \le \frac{3}{4}\,L(n)(t_n-n)^2 \to 0.$$

Also, for $x \geq x_0$, $\varepsilon > 0$,

$$l(x) \leq l\big(x + \varepsilon l(x)\big) \leq \frac{x + \varepsilon l(x)}{x}\, l(x) \sim l(x) \quad \text{as } x \to \infty$$

because $(l(x)/x) \searrow 0$ by (5).

It follows that $l(t_n) \sim l(n)$ and, since $h_1(t_n, n) \to 0$, that $(\Delta_n/a_n) \sim \sqrt{2\pi}\, l(n)$, and hence that $\sum_{k=1}^{\infty}(a_k/\Delta_k) = \infty$. This completes the proof. □

Proof of Theorem 4. We proceed as in [**14**, p. 52], [**15**, p. 486] using the notation introduced in Lemma 3. Suppose that N_0 is a large positive integer, and choose a nondecreasing function $h : [N_0, \infty) \to (-\infty, -\sigma)$ such that $x_n = -h(n)$ for $n \geq N_0$. Observe that, by Lemma 3, $-h(t) \searrow \sigma$ as $t \to \infty$. Let

$$c_n(t) := \frac{a_n e^{\lambda_n h(t)}}{a(-h(t))}.$$

Then $c_n(t) \geq 0$ and $\sum_{k=1}^{\infty} c_k(t) \equiv 1$ on $[N_0, \infty)$. Also, $c_n(t) \to 0$ as $t \to \infty$, because $a(x) \to \infty$ as $x \to \sigma+$ if $\sigma > -\infty$, and $a(x)e^{\lambda_n x} \geq a_{n+1}e^{(\lambda_n - \lambda_{n+1})x} \to \infty$ as $x \to \sigma = -\infty$. Next, let $\phi(t) := \sum_{k=N_0}^{[t]}(a_k/\Delta_k)$. Then, by Lemma 3,

$$\phi(t+1) - \phi(t) \sim \frac{\sqrt{L(t)}}{\sqrt{2\pi}} \to 0$$

and

$$\phi(t) \sim \frac{1}{\sqrt{2\pi}} \int_{N_0}^{t} \sqrt{L(v)}\, dv \to \infty \quad \text{as } t \to \infty.$$

Hence conditions (i) and (ii) of Lemma 2 are satisfied.

Since $h(t)$ is nondecreasing, and since $\Delta_n \leq a(x_k)e^{\lambda_n x_k} = \Delta_k e^{(\lambda_k - \lambda_n)h(k)}$ for $k, n \geq N_0$ by the definition of Δ_n, we obtain, for $t \geq M \geq N_0$, that

$$\begin{aligned}\phi(t) - \phi(M) &= \sum_{k=M+1}^{[t]} \frac{a_k}{\Delta_k} \leq \frac{1}{\Delta_M} \sum_{k=M+1}^{[t]} a_k e^{(\lambda_k - \lambda_M)h(k)} \\ &\leq \frac{1}{\Delta_M} \sum_{k=M+1}^{[t]} a_k e^{(\lambda_k - \lambda_M)h(t)} = \frac{e^{-\lambda_M h(t)}}{\Delta_M} \sum_{k=M+1}^{[t]} a_k e^{\lambda_k h(t)}.\end{aligned}$$

Hence, if $t > M \to \infty$ with $\phi(t) - \phi(M) \to \infty$, we get, since $h(t) \geq h(M)$, $\lambda_k \leq \lambda_M$, that

$$\begin{aligned}\sum_{k=1}^{M} c_k(t) &= \sum_{k=1}^{M} \frac{a_k e^{\lambda_k h(t)}}{a(-h(t))} = \frac{1}{a(-h(t))} \sum_{k=1}^{M} a_k e^{\lambda_k (h(M)+h(t)-h(M))} \\ &\leq \frac{e^{\lambda_M (h(t)-h(M))}}{a(-h(t))} \sum_{k=1}^{M} a_k e^{\lambda_k h(M)} \\ &= \frac{e^{-\lambda_M h(M)}}{a(-h(t)) e^{-\lambda_M h(t)}} \sum_{k=1}^{M} a_k e^{\lambda_k h(M)} \\ &\leq \frac{\Delta_M}{e^{-\lambda_M h(t)} \sum_{k=M+1}^{[t]} a_k e^{\lambda_k h(t)}} \leq \frac{1}{\phi(t) - \phi(M)} \to 0.\end{aligned}$$

Therefore condition (iii) of Lemma 2 is satisfied.

It follows from Lemma 3 that

$$\frac{a_n}{\Delta_n} \leq 1 \quad \text{for all } n \in \mathbf{N}, \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{a_k}{\Delta_k} = \infty.$$

Hence there exists $\tilde{l}(n) \in \mathbf{N}$ such that

$$\sum_{k=n+1}^{n+\tilde{l}(n)-1} \frac{a_k}{\Delta_k} < 1 \leq \sum_{k=n+1}^{n+\tilde{l}(n)} \frac{a_k}{\Delta_k} < 2 \quad \text{for } n \geq N_0.$$

In addition, by Lemma 3, we have that $\tilde{l}(n) \sim \sqrt{2\pi}\, l(n)$. For $N > t \to \infty$, we put $n_0 = [t]$, $n_{\nu+1} = n_\nu + \tilde{l}(n_\nu)$ for $\nu = 0, 1, 2, \ldots$, and $\alpha = \alpha(t, N)$ such that $n_{\alpha+1} > N \geq n_\alpha$. Then

$$\phi(N) - \phi(t) = \sum_{[t]+1}^{N} \frac{a_k}{\Delta_k} \leq \sum_{\nu=0}^{\alpha} \sum_{k=n_\nu+1}^{n_{\nu+1}} \frac{a_k}{\Delta_k} \leq 2(\alpha+1),$$

so that $\alpha \to \infty$ whenever $\phi(N) - \phi(t) \to \infty$. For $s \geq [t]$ we have that

$$\begin{aligned}
&\sum_{k=s+1}^{\infty} \frac{a_k}{a(-h(t))} e^{\lambda_k h(t)} \sum_{\nu=s+1}^{k} \frac{a_\nu}{\Delta_\nu} \\
&= \sum_{\nu=s+1}^{\infty} \frac{a_\nu}{\Delta_\nu} \sum_{k=\nu}^{\infty} \frac{a_k}{a(-h(t))} e^{\lambda_k h(t)} \\
&= \frac{1}{a(-h(t))} \sum_{\nu=s+1}^{\infty} \frac{a_\nu}{\Delta_\nu} e^{\lambda_\nu (h(t)-h(\nu))} \sum_{k=\nu}^{\infty} a_k e^{\lambda_k h(\nu)} \underbrace{e^{(\lambda_\nu - \lambda_k)(h(\nu)-h(t))}}_{\leq 1} \\
&\leq \frac{1}{a(-h(t))} \sum_{\nu=s+1}^{\infty} \frac{a_\nu}{\Delta_\nu} e^{\lambda_\nu h(t)} e^{-\lambda_\nu h(\nu)} a(-h(\nu)) \\
&= \sum_{\nu=s+1}^{\infty} \frac{a_\nu}{a(-h(t))} e^{\lambda_\nu h(t)}.
\end{aligned}$$

Using this inequality with $s = [t]$, we get that

$$\begin{aligned}
1 &\geq \sum_{\nu=[t]+1}^{\infty} \frac{a_\nu}{a(-h(t))} e^{\lambda_\nu h(t)} \geq \sum_{k=s+1}^{\infty} \frac{a_k}{a(-h(t))} e^{\lambda_k h(t)} \sum_{\nu=s+1}^{k} \frac{a_\nu}{\Delta_\nu} \\
&= \sum_{\mu=0}^{\infty} \sum_{k=n_\mu+1}^{n_{\mu+1}} \frac{a_k}{a(-h(t))} e^{\lambda_k h(t)} \sum_{\nu=n_0+1}^{k} \frac{a_\nu}{\Delta_\nu} \\
&\geq \sum_{\mu=0}^{\infty} \sum_{k=n_\mu+1}^{n_{\mu+1}} \frac{a_k}{a(-h(t))} e^{\lambda_k h(t)} \sum_{\nu=n_0+1}^{n_\mu} \frac{a_\nu}{\Delta_\nu} \\
&\geq \sum_{\mu=0}^{\infty} \mu \sum_{k=n_\mu+1}^{n_{\mu+1}} \frac{a_k}{a(-h(t))} e^{\lambda_k h(t)}.
\end{aligned}$$

These inequalities for $s = N$ yield that

$$\sum_{k=N}^{\infty} c_k(t)(\phi(k) - \phi(N))$$

$$= \sum_{k=N+1}^{\infty} \frac{a_k\, e^{\lambda_k h(t)}}{a(-h(t))} \sum_{\nu=N+1}^{k} \frac{a_\nu}{\Delta_\nu} \le \sum_{\nu=N+1}^{\infty} \frac{a_\nu}{a(-h(t))}\, e^{\lambda_\nu h(t)}$$

$$\le \sum_{\nu=\alpha}^{\infty} \sum_{k=n_\nu+1}^{n_{\nu+1}} \frac{a_k}{a(-h(t))}\, e^{\lambda_k h(t)} \le \sum_{\nu=\alpha}^{\infty} \frac{\nu}{\alpha} \sum_{k=n_\nu+1}^{n_{\nu+1}} \frac{a_k}{a(-h(t))}\, e^{\lambda_k h(t)}$$

$$\le \frac{1}{\alpha} \to 0 \quad \text{when } \phi(N) - \phi(t) \to \infty.$$

Thus condition (iv) of Lemma 2 is satisfied.

Finally, assumption (1*) of Theorem 4 implies that there exists $c > 0$ such that $s_{m+1} - s_n > -c$ for $n \le m \le n + \tilde{l}(n)$ for all $n \ge N_0$, so that $s(u) - s(t) > -c$ for $[t] + 1 \le [u] \le [t] + 1 + \tilde{l}([t])$, $t \ge N_0$, (note that $\tilde{l}(n) \sim \sqrt{2\pi l(n)}$). Let $n > t \ge N_0$, and put $t_0 = [t]$, $t_{\nu+1} = t_\nu + 1 + \tilde{l}(t_\nu)$ for $\nu = 0, 1, \dots, r+1$, with $t_{r+1} > u \ge t_r$. Then

$$s(u) - s(t) = \sum_{\nu=0}^{r-1} \left(s(t_{\nu+1}) - s(t_\nu)\right) + s(u) - s(t_r) > -(r+1)c,$$

and

$$r \le \sum_{\nu=0}^{r-1} \sum_{k=t_\nu+1}^{t_{\nu+1}} \frac{a_k}{\Delta_k} = \sum_{\nu=0}^{r-1} (\phi(t_{\nu+1}) - \phi(t_\nu)) = \phi(t_r) - \phi(t_0) = \phi(u) - \phi(t).$$

Hence, $s(u) - s(t) \ge -c(\phi(u) - \phi(t)) - c$, which shows that condition (v) of Lemma 2 is satisfied. Theorem 4 is now a consequence of Lemma 2. □

Theorem 5. *Assume* (C), *and suppose that* $L(x) \ge \delta > 0$ *on* $[x_0, \infty)$, *that* $s_n = O(1)(D_{\lambda,a})$, *and that condition* (2) *of Theorem* 3 *holds with some positive constant c. Then* $s_n = O(1)$.

The proof is based on the following estimates.

Lemma 4. *Assume* (C), *and suppose that* $L(x) \geq \delta > 0$ *on* $[x_0, \infty)$. *Let*

$$h(k,n) := g(n) - g(k) - \big(\lambda(n) - \lambda(k)\big)\frac{g'(n)}{\lambda'(n)},$$
$$\alpha_n := -h(n, n-1), \qquad \beta_n := -h(n-1, n)$$

as in Theorem 3, *and*

$$c_k(x) := \frac{a_k}{a(x)} e^{-\lambda_k x} \quad \textit{for } x > \sigma.$$

Then there exist $c_1 > 0$ *and* $x_1 > x_0$ *such that the following estimates hold for* $n \in \mathbf{N}$ *with* $n \geq x_1$:

(i) $h(k,n) \leq -\frac{1}{2}\gamma\delta|k-n|$ *for all* $k \in \mathbf{N}$, $k > x_0$, *where* $\gamma := \min\left(\frac{1}{3}, e^{-2\lambda'(x_0)/\lambda(x_0)}\right)$. *Also* $\sum_{k=x_1}^{\infty} e^{h(k,n)} \leq c_1$.

(ii) $\frac{1}{c_1} \leq \frac{a_n}{\Delta_n} \leq 1$ *with* Δ_n *defined as in Lemma* 3.

(iii)
$$\lambda(n+1) - \lambda(n) \geq \frac{1}{2}\big(\lambda(n) - \lambda(n-1)\big), \quad \textit{and}$$
$$\lambda(n-1) - \lambda(n-2) \geq \gamma\big(\lambda(n) - \lambda(n-1)\big).$$

(iv) $$\sum_{k=x_1}^{M-1} e^{h(k,n)} \leq c_1 e^{-(1/2)\delta\gamma(n-M+1)} \quad \textit{for } M \leq n+1,$$
and
$$\sum_{k=N}^{\infty}\sum_{\nu=N}^{k} e^{h(k,n)+\alpha_\nu} \leq c_1 e^{-\delta\gamma(N-n-1)} \quad \textit{for } N \geq n+1.$$

(v) $c_k(x) \leq c_1 e^{h(k,n)}$ *for all* $k \geq x_1$, *where* $x = -\frac{g'(n)}{\lambda'(n)}$.

(vi)
$$c_k(x) \leq c_1 e^{h(k,n)+f(t)-\alpha_n} \quad \textit{if } k \geq n,$$
$$c_k(x) \leq c_1 e^{h(k,n)-(1/2)(\alpha_n+\beta_n-f(t))} \quad \textit{if } k \geq n+1,$$
$$c_k(x) \leq c_1 e^{h(k,n-1)-\gamma f(t)} \quad \textit{if } x_1 \leq k \leq n-2,$$

$$c_k(x) \le c_1 e^{h(k,n-1)+\alpha_n - f(t)} \quad \textit{if } x_1 \le k \le n-1,$$

$$\textit{for } x = -\frac{g'(t)}{\lambda'(t)} \quad \textit{with } t \in [n-1, n], \quad \textit{and}$$

$$f(t) := \big(\lambda(n) - \lambda(n-1)\big)\left(\frac{g'(t)}{\lambda'(t)} - \frac{g'(n-1)}{\lambda'(n-1)}\right).$$

Proof. The first inequality in assertion (i) holds by [**1**, (10) and (11)]. Hence

$$\sum_{k=x_1}^{\infty} e^{h(k,n)} \le \sum_{k=x_1}^{\infty} e^{-(1/2)\gamma\delta|k-n|} \le 2\sum_{k=0}^{\infty} e^{-(1/2)\gamma\delta k} = c_1 < \infty,$$

which is the second inequality in assertion (i). For $x > \sigma$, $n \in \mathbf{N}$, we have that $a(x)e^{\lambda_n x} \ge a_n e^{\lambda_n x} e^{-\lambda_n x} = a_n$, so that $\Delta_n \ge a_n$. Moreover, since $a_n \sim e^{-g(n)}$, it follows that

$$\frac{\Delta_n}{a_n} \le \sum_{k=1}^{\infty} \frac{a_k}{a_n} e^{-(\lambda_n - \lambda_k)(g'(n)/\lambda'(n))} \sim \sum_{k=x_1}^{\infty} e^{h(k,n)} \le c_1 < \infty$$

by (i), and this establishes (ii).

By (C), we have that

$$\begin{aligned}
\lambda(n+1) - \lambda(n) &= \int_{n-1}^{n} \lambda'(u+1)\,du \\
&= \int_{n-1}^{n} \frac{(u+1)\lambda'(u+1)}{\lambda(u+1)} \frac{\lambda(u+1)}{u+1}\,du \\
&\ge \int_{n-1}^{n} \frac{u\lambda'(u)}{\lambda(u)} \frac{\lambda(u)}{u+1}\,du \ge \frac{n-1}{n}\int_{n-1}^{n} \lambda'(u)\,du \\
&\ge \frac{1}{2}\big(\lambda(n) - \lambda(n-1)\big),
\end{aligned}$$

and

$$\begin{aligned}\lambda(n-1)-\lambda(n-2) &= \int_{n-1}^{n} \lambda'(u-1)\,du \\ &\geq \int_{n-1}^{n} \frac{\lambda'(u)}{\lambda(u)}\,\lambda(u-1)\,du \qquad \left(\text{since } \frac{\lambda'}{\lambda}\searrow\right) \\ &= \int_{n-1}^{n} \lambda'(u)\exp\left(-\int_{u-1}^{u}\frac{\lambda'(w)}{\lambda(w)}\,dw\right)du \\ &\geq \gamma\big(\lambda(n)-\lambda(n-1)\big),\end{aligned}$$

because

$$\int_{u-1}^{u}\frac{\lambda'(w)}{\lambda(w)}\,dw \leq \frac{\lambda'(x_0)}{\lambda(x_0)} \leq -\log\gamma.$$

Hence (iii) holds.

The first inequality in (iv) follows directly from (i). In [**1**, Proof of Lemma 1] it is shown that

$$h(k,n) + \max_{n+1\leq j\leq k}\big(-h(j,j-1)\big) \leq -\delta\gamma(k-n-1) \quad \text{for } k \geq n+1. \tag{7}$$

Hence, if $N \geq n+1$, then

$$\begin{aligned}\sum_{k=N}^{\infty}\sum_{\nu=N}^{k} e^{h(k,n)+\alpha_\nu} &\leq \sum_{k=N}^{\infty}(k+1-N)e^{h(k,n)}\max_{n+1\leq j\leq k} e^{-h(j,j-1)} \\ &\leq \sum_{k=N}^{\infty}(k+1-N)e^{-\delta\gamma(k-n-1)} \\ &= e^{-\delta\gamma(N-n-1)}\sum_{k=0}^{\infty}(k+1)e^{-\gamma\delta k},\end{aligned}$$

which yields the second inequality in (iv).

Since $a_n \sim e^{-g(n)}$ and $a(x) \geq a_n e^{-\lambda_n x}$, we have that, for $x = -(g'(n)/\lambda'(n))$,

$$c_k(x) \leq c_1 e^{-g(k)+g(n)-(\lambda(n)-\lambda(k))(g'(n)/\lambda'(n))} = c_1 e^{h(k,n)},$$

which establishes (v).

By our assumptions and the notation in assertion (vi), we have that $n-1 \le t \le n$, $0 = f(n-1) \le f(t) \le f(n) = \alpha_n + \beta_n$, because $f'(\tau) = (g'(\tau)/\lambda'(\tau))' > 0$. Also, $a(x) \ge \max(a_n e^{-\lambda_n x}, a_{n-1} e^{-\lambda_{n-1} x})$. It follows that, for all $k \ge x_1$,

$$\begin{aligned} c_k(x) \le c_1 \exp\Big\{ &- g(k) + \lambda(k)\,\frac{g'(t)}{\lambda'(t)} \\ &+ \min\Big(g(n) - \lambda(n)\,\frac{g'(t)}{\lambda'(t)},\; g(n-1) - \lambda(n-1)\,\frac{g'(t)}{\lambda'(t)}\Big)\Big\}. \end{aligned} \tag{8}$$

First, if $k \ge n$, then

$$\begin{aligned} g(n-1) - g(k) &+ \big(\lambda(k) - \lambda(n-1)\big)\,\frac{g'(t)}{\lambda'(t)} \\ &= h(k,n) + f(t) - \alpha_n + \big(\lambda(k) - \lambda(n)\big)\Big(\frac{g'(t)}{\lambda'(t)} - \frac{g'(n)}{\lambda'(n)}\Big) \\ &\le h(k,n) + f(t) - \alpha_n, \end{aligned}$$

which yields the first inequality in (vi) by (8).

Next, if $k \ge n+1$, then by assertion (iii),

$$\begin{aligned} g(n) - g(k) &+ \big(\lambda(k) - \lambda(n)\big)\,\frac{g'(t)}{\lambda'(t)} \\ &= h(k,n) + \big(\lambda(n) - \lambda(k)\big)\Big(\frac{g'(n)}{\lambda'(n)} - \frac{g'(t)}{\lambda'(t)}\Big) \\ &\le h(k,n) + \big(\lambda(n) - \lambda(n+1)\big)\Big(\frac{g'(n)}{\lambda'(n)} - \frac{g'(t)}{\lambda'(t)}\Big) \\ &\le h(k,n) - \frac{1}{2}\,\big(\lambda(n) - \lambda(n-1)\big)\Big(\frac{g'(n)}{\lambda'(n)} - \frac{g'(t)}{\lambda'(t)}\Big) \\ &= h(k,n) - \frac{1}{2}\,\big(\alpha_n + \beta_n - f(t)\big), \end{aligned}$$

which yields the second inequality in (vi) by (8).

If $x_1 \le k \le n-2$, then by assertion (iii),

$$\begin{aligned} g(n-1) - g(k) + \big(\lambda(k) - \lambda(n-1)\big)\,\frac{g'(t)}{\lambda'(t)} \\ &= h(k,n-1) + \big(\lambda(k) - \lambda(n-1)\big)\left(\frac{g'(t)}{\lambda'(t)} - \frac{g'(n-1)}{\lambda'(n-1)}\right) \\ &\le h(k,n-1) + \big(\lambda(n-2) - \lambda(n-1)\big)\left(\frac{g'(t)}{\lambda'(t)} - \frac{g'(n-1)}{\lambda'(n-1)}\right) \\ &\le h(k,n-1) - \gamma\big(\lambda(n) - \lambda(n-1)\big)\left(\frac{g'(t)}{\lambda'(t)} - \frac{g'(n-1)}{\lambda'(n-1)}\right) \\ &= h(k,n-1) - \gamma f(t), \end{aligned}$$

so that the third inequality in (vi) holds by (8).

Finally, if $x_1 \le k \le n-1$, then

$$\begin{aligned} g(n) - g(k) + \big(\lambda(k) - \lambda(n)\big)\,\frac{g'(t)}{\lambda'(t)} \\ &= h(k,n-1) + \alpha_n - f(t) + \big(\lambda(k) - \lambda(n-1)\big)\left(\frac{g'(t)}{\lambda'(t)} - \frac{g'(n-1)}{\lambda'(n-1)}\right) \\ &\le h(k,n) + \alpha_n - f(t), \end{aligned}$$

which shows, by (8), that the last inequality in (vi) holds. □

Proof of Theorem 5. In this case Vijayaraghavan's theorem, i.e., Lemma 2, cannot be applied directly, but our method of proof is based essentially on the same techniques as Vijayaraghavan developed. Using the notation of Lemma 4 we model our proof on the proof of Theorem 238 in [**8**, pp. 308–312]. We suppose that $s_n \ne O(1)$, i.e., $\limsup_{n\to\infty} |s_n| = \infty$ and shall prove that this leads to a contradiction. Since

$$\sigma(x) = \sum_{k=1}^{\infty} s_k c_k(x) = O(1) \quad \text{as } x \to \sigma+,$$

the sequence $\{s_n\}$ cannot tend to either $+\infty$ or to $-\infty$. We write

$$\sigma_1(t) := \max_{1\le n\le t} s_n \quad \text{and} \quad \sigma_2(t) := \max_{1\le n\le t} (-s_n).$$

Hence $\sigma_1(t)$ and $\sigma_2(t)$ are nondecreasing and $\sigma_1(t) \to \infty$ or $\sigma_2(t) \to \infty$ as $t \to \infty$. There are two possibilities: either

(α) $\sigma_1(n) \geq \sigma_2(n)$ for infinitely many n, or

(β) $\sigma_1(n) < \sigma_2(n)$ for all sufficiently large n.

We consider these two possibilities in turn and show that each leads to a contradiction.

Case (α). Since condition (α) implies that $\sigma_1(n) \to \infty$ as $n \to \infty$ and, since $s_n \nrightarrow \infty$, so that $s_n \leq \tilde{c} < \infty$ for infinitely many n and some $\tilde{c} > 0$, it follows by our assumptions that there exists $H_0 > 0$ such that for all $H > H_0$ there is a *minimal* $M = M(H) \in \mathbf{N}$, $M \geq x_0$, with

$$s_M = \sigma_1(M) > 2H \quad \text{and} \quad \sigma_1(M) \geq \sigma_2(M), \tag{9}$$

and there is a *minimal* $N = N(H) > M$ with

$$s_N \leq \frac{1}{2} s_M. \tag{10}$$

Of course, $M(H) \nearrow \infty$ as $H \to \infty$. It follows from (9), (10) and the Tauberian condition (2) that

$$s_N - s_M = \sum_{k=M+1}^{N} (s_k - s_{k-1}) \geq -c \sum_{k=M+1}^{N} A_k,$$

and

$$s_N - s_M \leq -\frac{1}{2} s_M < -H.$$

Hence

$$\sum_{k=M+1}^{N} A_k \geq \frac{s_M}{2c} > \frac{H}{c} \to \infty \quad \text{as } H \to \infty. \tag{11}$$

For $x > \sigma$, let

$$\sigma(x) = \left(\sum_{k=1}^{M-1} + \sum_{k=M}^{N-1} + \sum_{k=N}^{\infty} \right) s_k c_k(x) =: \tau_1(x) + \tau_2(x) + \tau_3(x).$$

First, by (9),

$$\tau_1(x) = \sum_{k=1}^{M-1} s_k c_k(x) \geq -\sigma_2(M) \sum_{k=1}^{M-1} c_k(x) \geq -\sigma_1(M) \sum_{k=1}^{M-1} c_k(x).$$

Second, since $N > M$ is minimal such that (10) holds, we have that

$$\tau_2(x) = \sum_{k=M}^{N-1} s_k c_k(x) > \frac{1}{2}\, s_M \sum_{k=M}^{N-1} c_k(x).$$

Third, if $k \geq N$, then by (2), and since $s_{N-1} > (1/2)s_M$ by (10), we have that $s_k - s_{N-1} = \sum_{\nu=N}^{k}(s_\nu - s_{\nu-1}) \geq -c\sum_{\nu=N}^{k} A_\nu$, so that

$$\begin{aligned}\tau_3(x) &= \sum_{k=N}^{\infty} s_k c_k(x) = s_{N-1} \sum_{k=N}^{\infty} c_k(x) + \sum_{k=N}^{\infty} (s_k - s_{N-1}) c_k(x) \\ &> \frac{1}{2}\, s_M \sum_{k=N}^{\infty} c_k(x) - c \sum_{k=N}^{\infty} \sum_{\nu=N}^{k} A_\nu c_k(x).\end{aligned}$$

Altogether we have shown that

$$\sigma(x) \geq -s_M \sum_{k=1}^{M-1} c_k(x) + \frac{1}{2}\, s_M \sum_{k=M}^{\infty} c_k(x) - c \sum_{k=N}^{\infty} \sum_{\nu=N}^{k} A_\nu c_k(x),$$

that is,

$$\sigma(x) \geq s_M \left(\frac{1}{2} - \frac{3}{2} \sum_{k=1}^{M-1} c_k(x) \right) - c \sum_{k=N}^{\infty} \sum_{\nu=N}^{k} A_\nu c_k(x). \qquad (12)$$

We consider two cases.

Case (α1). $\limsup_{H\to\infty}(N(H) - M(H)) = \infty$.

Let $n = n(m) := [(N(H_m) + M(H_m))/2]$ such that $N - M = N(H_m) - M(H_m) \to \infty$ as $m \to \infty$. Then $N - n \to \infty$ and $n - M \to \infty$ as $m \to \infty$. Put $x = x_n := -(g'(n)/\lambda'(n))$. Then, by parts (iv) and

(v) of Lemma 4 and, using that $c_k(t) \to 0$ as $t \to \sigma+$, we get that, as $m \to \infty$,

$$\sum_{k=1}^{M-1} c_k(x) \le o(1) + c_1 \sum_{k=x_1}^{M-1} e^{h(k,n)} \le o(1) + c_1^2 e^{-\delta\gamma(n-M+1)/2} \to 0,$$

and

$$\sum_{k=N}^{\infty} \sum_{\nu=N}^{k} A_\nu c_k(x) \le c_1 \sum_{k=N}^{\infty} \sum_{\nu=N}^{k} e^{h(k,n)+\alpha_\nu} \le c_1^2 e^{-\delta\gamma(N-n-1)} \to 0.$$

Therefore, by (12) and (9), $\sigma(x) = \sigma(x_n) \to \infty$, contradicting our assumption that $\sigma(x) = O(1)$.

Case (α2). $N(H) - M(H) \le c_2 < \infty$ for all $H > H_0$ and some $c_2 > 0$.

By (11), there exists $n \in \{M+1, \dots, N\}$ such that

$$(11') \qquad A_n \ge \frac{s_M}{2cc_2} > \frac{H}{cc_2} \to \infty \quad \text{as } H \to \infty.$$

Now choose $t = t_n \in [n-1, n]$ such that

$$(13) \qquad \begin{aligned} 0 = f(n-1) \le f(t) &= \min\left(\alpha_n, \frac{1}{2} \log s_M\right) \\ &\le \alpha_n + \beta_n = f(n), \end{aligned}$$

so that $f(t_n) \to \infty$, and put $x = x_n := -(g'(t_n)/\lambda'(t_n))$. Then, by parts (i), (iv) and (vi) of Lemma 4 and (13),

$$\begin{aligned} \sum_{k=1}^{M-1} c_k(x) \le o(1) + \sum_{k=x_1}^{n-2} c_k(x) &\le o(1) + c_1 \sum_{k=x_1}^{n-2} e^{h(k,n-1)-\gamma f(t)} \\ &\le o(1) + c_1^2 e^{-\gamma f(t_n)} \to 0, \end{aligned}$$

and

$$\sum_{k=N}^{\infty}\sum_{\nu=N}^{k} A_\nu c_k(x)$$
$$\le \sum_{k=n}^{\infty}\sum_{\nu=n}^{k} A_\nu c_k(x) = A_n \sum_{k=n}^{\infty} c_k(x) + \sum_{k=n+1}^{\infty}\sum_{\nu=n+1}^{k} A_\nu c_k(x)$$
$$\le c_1 \sum_{k=n}^{\infty} e^{\alpha_n + h(k,n) + f(t) - \alpha_n} + c_1 \sum_{k=n+1}^{\infty}\sum_{\nu=n+1}^{k} e^{h(k,n)+f(t)-\alpha_n+\alpha_\nu}$$
$$\le c_1 e^{f(t)} \sum_{k=n}^{\infty} e^{h(k,n)} + c_1 \sum_{k=n+1}^{\infty}\sum_{\nu=n+1}^{k} e^{h(k,n)+\alpha_\nu}$$
$$\le c_1^2(\sqrt{s_M}+1) = o(s_M).$$

Therefore, by (12) and (9), $\sigma(x) = \sigma(x_n) \to \infty$, contradicting the hypothesis that $\sigma(x) = O(1)$. Thus case (α) leads to a contradiction.

Case (β). $\sigma_1(n) < \sigma_2(n)$ for all $n \ge N_0$. This implies that $\sigma_2(n) \to \infty$ as $n \to \infty$ and, since $s_n \nrightarrow -\infty$ so that $s_n \ge -\tilde{c} > -\infty$ for infinitely many n and some $\tilde{c} > 0$, it follows, by a similar argument to the one used in case (α), that there exists $H_0 > 0$ such that for all $H > H_0$ there is a *minimal* $N = N(H) \in \mathbf{N}$ with

$$(14) \qquad s_N = -\sigma_2(N) < -2H \quad \text{and} \quad \sigma_1(n) < \sigma_2(n) \quad \text{for all } n \ge N,$$

and there is a *maximal* $M = M(H) < N$, $M \ge x_0$, with

$$(15) \qquad s_M \ge -\frac{1}{2}\sigma_2(N) = \frac{1}{2}s_N,$$

so that $s_n < s_N/2$ for $M < n \le N$, by condition (β). Of course, $N(H) \nearrow \infty$ as $H \to \infty$. It follows from (14), (15), and the Tauberian condition (2) that

$$s_N - s_M = \sum_{k=M+1}^{N} (s_k - s_{k-1}) \ge -c \sum_{k=M+1}^{N} A_k,$$

and

$$s_N - s_M \leq \frac{1}{2}\, s_N < -H.$$

Hence

$$\sum_{k=M+1}^{N} A_k \geq -\frac{s_N}{2c} > \frac{H}{c} \to \infty \quad \text{as } H \to \infty. \tag{16}$$

For $x > \sigma$, let

$$\sigma(x) = \Big(\sum_{k=1}^{M} + \sum_{k=M+1}^{N} + \sum_{k=N+1}^{\infty}\Big) s_k c_k(x) =: \tau_1(x) + \tau_2(x) + \tau_3(x).$$

First, by (14),

$$\tau_1(x) = \sum_{k=1}^{M} s_k c_k(x) \leq \sigma_1(N) \sum_{k=1}^{M} c_k(x) \leq \sigma_2(N) \sum_{k=1}^{M} c_k(x).$$

Second, since $M < N$ is maximal such that (15) holds, we have that

$$\tau_2(x) = \sum_{k=M+1}^{N} s_k c_k(x) \leq -\frac{1}{2}\, \sigma_2(N) \sum_{k=M+1}^{N} c_k(x) = \frac{1}{2}\, s_N \sum_{k=M+1}^{N} c_k(x).$$

Third, if $k \geq N+1$, then by (2) $s_k - s_N \geq -c \sum_{\nu=N+1}^{k} A_\nu$, so that

$$\sigma_2(k) = \max_{1 \leq \nu \leq k} (-s_\nu) \leq -s_N + c \sum_{\nu=N+1}^{k} A_\nu,$$

because $\sigma_2(N) = -s_N$ by (15). Hence, by (14),

$$\begin{aligned}\tau_3(x) = \sum_{k=N+1}^{\infty} s_k c_k(x) &\leq \sum_{k=N+1}^{\infty} \sigma_1(k) c_k(x) \leq \sum_{k=N+1}^{\infty} \sigma_2(k) c_k(x) \\ &\leq -s_N \sum_{k=N+1}^{\infty} c_k(x) + c \sum_{k=N+1}^{\infty} \sum_{\nu=N+1}^{k} A_\nu c_k(x).\end{aligned}$$

Altogether we have shown that

$$\sigma(x) \le -s_N \sum_{k=1}^{M} c_k(x) + \frac{1}{2} s_N \sum_{k=M+1}^{N} c_k(x) - s_N \sum_{k=N+1}^{\infty} c_k(x) + c \sum_{k=N+1}^{\infty} \sum_{\nu=N+1}^{k} A_\nu c_k(x),$$

that is

$$\sigma(x) \le s_N\left(\frac{1}{2} - \frac{3}{2}\sum_{k=1}^{M} c_k(x) - \frac{3}{2}\sum_{k=N+1}^{\infty} c_k(x)\right) + c \sum_{k=N+1}^{\infty} \sum_{\nu=N+1}^{k} A_\nu c_k(x). \tag{17}$$

Again we consider two cases.

Case (β1). $\limsup_{H\to\infty}(N(H) - M(H)) = \infty$.

Let $n = n(m) := [(N(H_m) + M(H_m))/2]$ such that $N - M = N(H_m) - M(H_m) \to \infty$ as $m \to \infty$. Then $N - n \to \infty$ and $n - M \to \infty$ as $m \to \infty$. Put $x = x_n := -(g'(n)/\lambda'(n))$. Then, by parts (iv) and (v) of Lemma 4, and using that $c_k(t) \to 0$ as $t \to \sigma+$, we get that, as $m \to \infty$,

$$\sum_{k=1}^{M} c_k(x) \le o(1) + c_1 \sum_{k=x_1}^{M} e^{h(k,n)} \le o(1) + c_1^2 e^{-\delta\gamma(n-M)/2} \to 0,$$

and (since $A_k \ge 1$ for all k)

$$\sum_{k=N+1}^{\infty} c_k(x) \le \sum_{k=N+1}^{\infty} \sum_{\nu=N+1}^{k} A_\nu c_k(x) \le c_1 \sum_{k=N+1}^{\infty} \sum_{\nu=N+1}^{k} e^{h(k,n)+\alpha_\nu} \le c_1^2 e^{-\delta\gamma(N-n)} \to 0.$$

Therefore, by (17) and (14), $\sigma(x) = \sigma(x_n) \to -\infty$, contradicting our assumption that $\sigma(x) = O(1)$.

Case ($\beta 2$). $N(H) - M(H) \le c_2 < \infty$ for all $H > H_0$ and some $c_2 > 0$. By (16), there exists $n \in \{M+1, \dots, N\}$ such that

$$A_n \ge -\frac{s_N}{2cc_2} > \frac{H}{cc_2} \to \infty \quad \text{as } H \to \infty. \tag{16'}$$

Now choose $t = t_n \in [n-1, n]$ such that

$$0 = f(n-1) \le f(t) = \alpha_n + \frac{1}{2}\beta_n \le \alpha_n + \beta_n = f(n), \tag{18}$$

and put $x = x_n := -(g'(t_n)/\lambda'(t_n))$. Then, by parts (iv) and (vi) of Lemma 4, (18) and (16′),

$$\begin{aligned}\sum_{k=1}^{M} c_k(x) &\le o(1) + \sum_{k=x_1}^{n-1} c_k(x) \le o(1) + c_1 \sum_{k=x_1}^{n-1} e^{h(k,n-1)+\alpha_n - f(t)} \\ &\le o(1) + c_1 e^{-(\beta_n/2)} \sum_{k=x_1}^{n-1} e^{h(k,n-1)} \le o(1) + c_1^2 A_n^{-1/2} \to 0,\end{aligned}$$

and

$$\begin{aligned}\sum_{k=N+1}^{\infty} c_k(x) &\le \sum_{k=N+1}^{\infty} \sum_{\nu=N+1}^{k} A_\nu c_k(x) \le \sum_{k=n+1}^{\infty} \sum_{\nu=n+1}^{k} A_\nu c_k(x) \\ &\le c_1 \sum_{k=n+1}^{\infty} \sum_{\nu=n+1}^{k} e^{h(k,n)-(\alpha_n+\beta_n-f(t))/2+\alpha_\nu} \\ &= c_1 e^{-(\beta_n/4)} \sum_{k=n+1}^{\infty} \sum_{\nu=n+1}^{k} e^{h(k,n)+\alpha_\nu} \le c_1^2 A_n^{-1/4} \to 0.\end{aligned}$$

Therefore, by (17) and (14), $\sigma(x) = \sigma(x_n) \to -\infty$, contradicting the hypothesis that $\sigma(x) = O(1)$. Thus case (β) leads to a contradiction, and this completes the proof of Theorem 5. □

REFERENCES

1. J. Beurer, D. Borwein and W. Kratz, *Two Tauberian theorems for Dirichlet series methods of summability*, Acta Sci. Math. (Szeged) **65** (1999), 143–172.

2. N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular variation*, Cambridge Univer. Press, Cambridge, 1987.

3. D. Borwein, *Tauberian and other theorems concerning Dirichlet's series with non- negative coefficients*, Math. Proc. Cambridge Philos. Soc. **102** (1987), 517-532.

4. ———, *Tauberian and other theorems concerning Laplace transforms and Dirichlet series*, Arch. Math. (Basel) **53** (1989), 352–362.

5. D. Borwein and W. Kratz, *An O-Tauberian theorem and a high indices theorem for power series methods of summability*, Math. Proc. Cambridge Philos. Soc. **15** (1994), 365–375.

6. ———, *A high indices Tauberian theorem*, Acta Sci. Math. (Szeged) **64** (1998), 143–149.

7. D. Gaier, *Der allgemeine Lückensatz für das Borel- Verfahren*, Math. Z. **88** (1965), 410–417.

8. G.H. Hardy, *Divergent series*, Oxford University Press, London, 1949.

9. G.H. Hardy and J.E. Littlewood, *A further note on the converse of Abel's theorem*, Proc. London Math. Soc. (2) **25** (1926), 219–236.

10. A.E. Ingham, *On the 'High-Indices Theorem' of Hardy and Littlewood*, Quart. J. Math. Oxford **8** (1937), 1–7.

11. A. Jakimovski and H. Tietz, *Regularly varying functions and power series methods*, J. Math. Anal. Appl. **73** (1980), 65–84.

12. A. Jakimovski, W. Meyer-König and K. Zeller, *Power series methods of summability: positivity and gap perfectness*, Trans. Amer. Math. Soc. **266** (1981), 309–317.

13. M.L. Kales, *Tauberian theorems related to Borel and Abel summability*, Duke Math. J. **3** (1937), 647–666.

14. R. Kiesel, *Taubersätze und starke Gesetze für Potenzreihenverfahren*, Dissertation, Ulm, 1990.

15. R. Kiesel and U. Stadtmüller, *Tauberian theorems for general power series methods*, Math. Proc. Cambridge Philos. Soc. **110** (1991), 483–490.

16. W. Kratz and U. Stadtmüller, *Tauberian theorems for J_p-summability*, J. Math. Anal. Appl. **139** (1989), 362–371.

17. ———, *O-Tauberian theorems for J_p-methods with rapidly increasing weights*, J. London Math. Soc. (2) **41** (1990), 489–502.

18. ———, *Tauberian theorems for Borel-type methods of summability*, Arch. Math. (Basel) **55** (1990), 465–474.

19. V.K. Krishnan, *Gap Tauberian theorem for logarithmic summability (L)*, Quart. J. Math. Oxford **30** (1970), 77–87.

20. E. Landau, *Über die Konvergenz einiger Klassen von unendlichen Reihen am Rande des Konvergenzgebietes*, Mh. Math. **18** (1907), 8–28.

21. R. Schmidt, *Die Umkehrsätze des Borelschen Summierungsverfahrens*, Schriften Königsberg **1** (1925), 205–256.

22. ———, *Über divergente Folgen und lineare Mittelbildungen*, Math. Z. **22** (1925), 87–152.

23. U. Stadtmüller, *Tauberian theorems for weighted Laplace transforms*, Analysis **13** (1993), 159–183.

24. U. Stadtmüller and R. Trautner, *Tauberian theorems for Laplace transforms*, J. Reine Angew. Math. **311** (1979), 283–290.

25. H. Tietz and R. Trautner, *Tauber-Sätze für Potenzreihenverfahren*, Arch. Math. (Basel) **50** (1988), 164–174.

26. T. Vijayaraghavan, *A Tauberian theorem*, J. London Math. Soc. (1) **1** (1926), 113–120.

27. ———, *A theorem concerning the summability of series by Borel's method*, Proc. London Math. Soc. (2) **27** (1928), 316–326.

Department of Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B7
E-mail address: dborwein@uwo.ca

Abteilung Mathematik, Universität Ulm, D-89069 Ulm, Germany
E-mail address: kratz@mathematik.uni-ulm.de

Abteilung Mathematik, Universität Ulm, D-89069 Ulm, Germany
E-mail address: stamue@mathematik.uni-ulm.de