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A high indices Tauberian theorem

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Abstract. We prove a Tauberian theorem concerning the summability method $D_{\lambda,a}$ based on the Dirichlet series $\sum a_n e^{-\lambda_n x}$ with $a_{n+1} \sim a_n > 0$ when the sequence (λ_n) satisfies the 'high indices' condition $\lambda_{n+1} > c\lambda_n \geq 0$ with any $c > 1$.

1. Introduction

Suppose throughout that (λ_n) is a strictly increasing unbounded sequence of real numbers with $\lambda_1 \geq 0$, and that (a_n) is a sequence of positive numbers. Suppose also that

$$A_n := \sum_{k=1}^{n} a_k \to \infty,$$

and define

$$a(x) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}$$

whenever this Dirichlet series converges.

Let $s, s_1, s_2, \ldots$ be complex numbers, and define

$$\sigma_n := \frac{1}{A_n} \sum_{k=1}^{n} a_k s_k \qquad \text{and} \qquad \sigma(x) := \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x}.$$

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The weighted mean summability method M_a and the Dirichlet series summability method $D_{\lambda,a}$ (see [1]) are defined as follows:

$$s_n \to \begin{cases} s(M_a) & \text{if } \sigma_n \to s; \\ s(D_{\lambda,a}) & \text{if } \sigma(x) \text{ exists for } x > 0 \text{ and } \sigma(x) \to s \text{ as } x \to 0+. \end{cases}$$

It is known (see [1]) that, since $A_n \to \infty$, both methods are regular (i.e., $s_n \to s$ implies $s_n \to s(M_a)$ and $s_n \to s(D_{\lambda,a})$), and that $s_n \to s(M_a)$ implies $s_n \to s(D_{\lambda,a})$.

The purpose of this paper is to prove the following result:

Theorem 1. *Suppose that the sequence (a_n) satisfies the condition*

$$a_{n+1} \sim a_n, \tag{1}$$

and that the sequence (λ_n) satisfies the high indices condition

$$\lambda_{n+1} > c\lambda_n \quad \text{with} \quad c > 1. \tag{2}$$

Suppose also that $s_n \to s(D_{\lambda,a})$, and that the Tauberian condition

$$s_{n+1} - s_n = O\left(\frac{a_n}{A_n}\right) \tag{3}$$

holds. Then $s_n \to s$.

Remark. Observe that conditions (1) and (2) imply that, for $x > 0$,

$$\frac{a_{n+1}e^{-\lambda_{n+1}x}}{a_n e^{-\lambda_n x}} \le \frac{a_{n+1}}{a_n} e^{-(c-1)\lambda_n x} \to 0,$$

from which it follows that the Dirichlet series $a(x)$ converges for all $x > 0$.

It is interesting to compare Theorem 1 with Theorem 114 in [3], namely:

Theorem H1. *Suppose that (2) holds, and that*

$$\sum_{n=1}^{\infty} (s_{n+1} - s_n)e^{-\lambda_n x} \to s \quad as \quad x \to 0+.$$

Then $s_n \to s$.

A known Tauberian result concerning sequences (λ_n) not satisfying the high indices condition (2) is the following theorem due to Borwein [2, Theorem 6] (see also Tietz [6, Satz 3.9] for the case $\lambda_n = n$):

Theorem B1. *Suppose that*

$$\lambda_{n+1} \sim \lambda_n,$$

$$\frac{A_m}{A_n} \to 1 \quad when \quad \frac{\lambda_m}{\lambda_n} \to 1,\ m > n \to \infty,$$

$$\liminf (s_m - s_n) \ge 0 \quad when \quad \frac{A_m}{A_n} \to 1,\ m > n \to \infty,$$

and that $s_n \to s(D_{\lambda,a})$. Then $s_n \to s$.

We won't use the above two results, but will use another known Tauberian result which follows immediately from Theorem 67 in [3]:

Theorem H2. *If (3) holds and $s_n \to s(M_a)$, then $s_n \to s$.*

We will also use the following Tauberian theorem due to Borwein [1, Theorem 2]:

Theorem B2. *Let $s_n \to s(D_{\lambda,a})$, let $s_n > -H$ where H is a constant, and let*

$$\lim_{x \to 0+} \frac{a(mx)}{a(x)} = \alpha_m > 0 \quad for \quad m = 2 \quad and \quad m = 3.$$

Then $s_n \to s(M_a)$.

For other Tauberian theorems of the type $D_{\lambda,a} \Rightarrow M_a$ see [1, Theorem 3], [2, Theorem 7], and (for the case $\lambda_n = n$) [5, Korollar 4.2].

2. Auxiliary results

Lemma 1. *Suppose (x_n) is a non-increasing sequence of positive numbers such that $\sigma(x_n)$ exists. Suppose also that (1) and (2) hold, and that*

$$|s_{n+1} - s_n| \le c_1 \frac{a_n}{\delta_n} \quad with \quad \delta_n := a(x_n)e^{\lambda_n x_n},$$

where c_1 is a positive constant. Then $|\sigma(x_n) - s_n| \le c_1$.

Proof. (Cf. the proof of [4, Theorem1].) We have

$$|\sigma(x_n) - s_n| = \left| \frac{1}{a(x_n)} \sum_{k=1}^{\infty} a_k (s_k - s_n) e^{-\lambda_k x_n} \right| \le \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 := \frac{1}{a(x_n)} \sum_{k=1}^{n-1} a_k |s_k - s_n| e^{-\lambda_k x_n}, \qquad \Sigma_2 := \frac{1}{a(x_n)} \sum_{k=n+1}^{\infty} a_k |s_k - s_n| e^{-\lambda_k x_n}.$$

We have

$$\begin{aligned} \Sigma_1 :&\le \frac{1}{a(x_n)} \sum_{k=1}^{n-1} a_k e^{-\lambda_k x_n} \sum_{j=k}^{n-1} c_1 \frac{a_j}{\delta_j} \\ &= \frac{c_1}{a(x_n)} \sum_{j=1}^{n-1} \frac{a_j}{\delta_j} \frac{e^{-\lambda_j x_n}}{e^{-\lambda_j x_j}} \sum_{k=1}^{j} a_k e^{-\lambda_k x_j} e^{(x_j - x_n)(\lambda_k - \lambda_j)} \\ &\le \frac{c_1}{a(x_n)} \sum_{j=1}^{n-1} a_j e^{-\lambda_j x_n} \frac{1}{a(x_j)} \sum_{k=1}^{j} a_k e^{-\lambda_k x_j} \le \frac{c_1}{a(x_n)} \sum_{j=1}^{n-1} a_j e^{-\lambda_j x_n}, \end{aligned}$$

since $\lambda_k \le \lambda_j$ and $x_j \ge x_n$ when $k \le j < n$. Next we have

$$\begin{aligned} \Sigma_2 :&\le \frac{1}{a(x_n)} \sum_{k=n+1}^{\infty} a_k e^{-\lambda_k x_n} \sum_{j=n}^{k-1} c_1 \frac{a_j}{\delta_j} \\ &= \frac{c_1}{a(x_n)} \sum_{j=n}^{\infty} \frac{a_j}{\delta_j} \frac{e^{-\lambda_j x_n}}{e^{-\lambda_j x_j}} \sum_{k=j+1}^{\infty} a_k e^{-\lambda_k x_j} e^{(x_j - x_n)(\lambda_k - \lambda_j)} \\ &\le \frac{c_1}{a(x_n)} \sum_{j=n}^{\infty} a_j e^{-\lambda_j x_n} \frac{1}{a(x_j)} \sum_{k=j+1}^{\infty} a_k e^{-\lambda_k x_j} \le \frac{c_1}{a(x_n)} \sum_{j=n}^{\infty} a_j e^{-\lambda_j x_n}, \end{aligned}$$

since $\lambda_k \ge \lambda_j$ and $x_j \le x_n$ when $n \le j < k$. Hence

$$\Sigma_1 + \Sigma_2 \le \frac{c_1}{a(x_n)} \sum_{j=1}^{\infty} a_j e^{-\lambda_j x_n} = c_1,$$

which yields the desired conclusion. ∎

Lemma 2. *Suppose that (1) and (2) hold. Then*
(i) $A_{n+1} \sim A_n$;
(ii) *for all sufficiently large* n, $(1/e)A_n \le a(1/\lambda_n) \le c_2 A_n$, *where* $c_2 := 1 + \sum_{k=1}^{\infty} e^{k - c^k} < \infty$ *with* $c > 1$ *from (2); and*
(iii) *for all* $t > 0$, $(a(tx)/a(x)) \to 1$ *as* $x \to 0+$.

Proof. (i) It follows from (1) and the regularity of M_a that

$$\frac{A_{n+1}}{A_n} = \frac{1}{A_n} \left(a_1 + \sum_{k=1}^{n} \frac{a_{k+1}}{a_k} a_k \right) \to 1.$$

Hence, for sufficiently large n and all $k \ge 0$,

$$\frac{A_{k+n}}{A_n} = \prod_{j=0}^{k-1} \frac{A_{n+j+1}}{A_{n+j}} \le e^k.$$

(ii) Next we have, for sufficiently large n,

$$\begin{aligned} \frac{1}{e} A_n &\le \sum_{k=1}^{n} a_k e^{-\lambda_k/\lambda_n} \le a\left(\frac{1}{\lambda_n}\right) \le A_n \left(1 + \sum_{k=n+1}^{\infty} \frac{a_k}{A_n} e^{-\lambda_k/\lambda_n}\right) \\ &\le A_n \left(1 + \sum_{k=1}^{\infty} \frac{A_{k+n}}{A_n} e^{-\lambda_{k+n}/\lambda_n}\right) \le A_n \left(1 + \sum_{k=1}^{\infty} e^k e^{-c^k}\right) = c_2 A_n, \end{aligned}$$

since $(A_{k+n}/A_n) \le e^k$ and $(\lambda_{k+n}/\lambda_n) \ge c^k$.

(iii) Since $a(x)$ decreases as x increases and $c > 1$, we have, by (1), (2) and the regularity of $D_{\lambda,a}$, that, for $x > 0$,

$$\begin{aligned} 1 \ge \frac{a(cx)}{a(x)} &= \frac{1}{a(x)} \sum_{k=1}^{\infty} a_k e^{-c\lambda_k x} \ge \frac{1}{a(x)} \sum_{k=1}^{\infty} a_k e^{-\lambda_{k+1} x} \\ &= \frac{1}{a(x)} \sum_{k=2}^{\infty} \frac{a_{k-1}}{a_k} a_k e^{-\lambda_k x} \to 1 \quad \text{as} \quad x \to 0+, \end{aligned}$$

which implies (iii). ∎

3. Proof of Theorem 1

Put $x_n := 1/\lambda_n$ whence, in the notation of Lemma 1, $\delta_n = ea(x_n)$. Hence, by parts (i) and (ii) of Lemma 2,

$$s_{n+1} - s_n = O\left(\frac{a_n}{A_n}\right) = O\left(\frac{a_n}{\delta_n}\right),$$

and $\sigma(x_n) \to s$ because $s_n \to s(D_{\lambda,a})$. Therefore, by Lemma 1, the sequence (s_n) is bounded, and so, by Lemma 2(iii) and Theorem B2, $s_n \to s(M_a)$. It follows, by Theorem H2, that $s_n \to s$. ∎

Remark. The order of magnitude in our Tauberian condition (3) is best possible in the following sense: if (γ_n) is any sequence of positive numbers with $\gamma_n \to \infty$, then there is a divergent sequence (s_n) such that

$$s_{n+1} - s_n = O\left(\gamma_n \frac{a_n}{A_n}\right) \quad \text{and} \quad s_n \to 0(M_a) \quad (\text{and hence} \quad s_n \to 0(D_{\lambda,a})).$$

This is certainly well-known and can be established by elementary means, but we have not seen a published proof.

4. Examples

In the following two examples we consider the case $s = 0$ of Theorem 1, and assume that each sequence (λ_n) satisfies (2). The order of magnitude of $a(x)$ as $x \to 0+$ is easily determined from the observation that, by parts (i) and (ii) of Lemma 2,

$$\frac{1}{e}A_n \le a(x) \le c_2 A_{n+1} \sim c_2 A_n \quad \text{when} \quad \frac{1}{\lambda_{n+1}} \le x \le \frac{1}{\lambda_n} \quad \text{and} \quad n \to \infty.$$

(i) Suppose that $\lambda_n \ge \lambda^n$ for some $\lambda > 1$ and all sufficiently large n, and that $a_k = k^\alpha$ for some $\alpha > -1$. Then

$$A_n \sim \frac{n^{1+\alpha}}{1+\alpha}$$

and, for $x > 0$ and some positive constant M,

$$a(x) \le M\left(\log\left(\frac{1}{x}\right)\right)^{-1-\alpha}.$$

Thus, if

$$\left(\log\left(\frac{1}{x}\right)\right)^{-1-\alpha} \sum_{k=1}^{\infty} s_k k^\alpha e^{-\lambda_k x} \to 0 \quad \text{as} \quad x \to 0+$$

and $s_{n+1} - s_n = O(1/n)$, then $s_n \to 0$. By putting $t = e^{-x}$, we can transform this into the following result involving power series with 'Hadamard gaps': if

$$\left(\log\left(\frac{1}{1-t}\right)\right)^{-1-\alpha} \sum_{k=1}^{\infty} s_k k^\alpha t^{\lambda_k} \to 0 \quad \text{as} \quad t \to 1-$$

and $s_{n+1} - s_n = O(1/n)$, then $s_n \to 0$.

(ii) Suppose that $\lambda_n \ge \exp(2^n)$ for all sufficiently large n. As above, we now have that if

$$\frac{1}{\log\log\left(\frac{1}{1-t}\right)} \sum_{k=1}^{\infty} s_k t^{\lambda_k} \to 0 \quad \text{as} \quad t \to 1-$$

and $s_{n+1} - s_n = O(1/n)$, then $s_n \to 0$.

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