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A GENERALIZATION OF HARDY'S INEQUALITY WITH APPLICATIONS

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ABSTRACT: A unifying theorem concerning general inequalities of the Hardy type is established, and it is shown how its specializations can be used to prove many inequalities, some of which appear to be new.

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1. INTRODUCTION AND THE MAIN THEOREM

Suppose throughout that $(s_n)_{n\geq 0}$ is a sequence of complex numbers, and that $s(x)$ is a complex, Borel measurable function on $[0,\infty)$. Suppose also that, unless otherwise stated, $\alpha(x)$ is a real function of bounded variation on $[0,1]$, that $\beta(x)$ is a real function of bounded variation on $[0,\infty)$, and that $1<r<\infty$ and $1/r+1/r'=1$. Hardy's inequality Hardy et al [7, #326] is:

$$\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{m=1}^{n}s_m\right|^r \leq \left(\frac{r}{r-1}\right)^r\sum_{m=1}^{\infty}|s_m|^r,$$

and there have been many generalizations of it. We shall establish the following unifying theorem concerning general inequalities of the Hardy type, and then show how it and its variants can be specialized to prove many inequalities, some of which hitherto have been proved by diverse ad hoc methods, and some of which we believe to be new.

Theorem 1. *Let the functions $g(x), h(x), f_{mn}(x) (m,n=0,1,\dots)$ be non-negative and Borel measurable on $[0,1]$. Suppose there exist positive numbers p_n, q_n $(n=0,1,\dots)$ such that*

$$\sum_{n=0}^{\infty} f_{mn}(x) \leq p_m g(x) \quad \text{for} \quad 0\leq x\leq 1, \quad m=0,1,\dots, \tag{1.1}$$

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and

$$\sum_{m=0}^{\infty} f_{mn}(x) \le q_n h(x) \quad \text{for} \quad 0 \le x \le 1, \quad n = 0, 1, \ldots . \tag{1.2}$$

Then

$$\sum_{n=0}^{\infty} q_n{}^{1-r} \left| \sum_{m=0}^{\infty} s_m \int_0^1 f_{mn}(x) d\alpha(x) \right|^r \le \left(\int_0^1 g(x)^{1/r} h(x)^{1/r'} |d\alpha(x)| \right)^r \sum_{m=0}^{\infty} p_m |s_m|^r , \tag{1.3}$$

and

$$\sum_{m=0}^{\infty} p_m{}^{1-r} \left| \sum_{n=0}^{\infty} s_n \int_0^1 f_{mn}(x) d\alpha(x) \right|^r \le \left(\int_0^1 g(x)^{1/r'} h(x)^{1/r} |d\alpha(x)| \right)^r \sum_{m=0}^{\infty} q_m |s_m|^r . \tag{1.4}$$

Remarks about Theorem 1.

(i) Any integral of the form $\int_0^1 w(x) d\alpha(x)$ is to be interpreted to be the Lebesgue-Stieltjes integral $\int_{[0,1]} w(x) d\alpha(x)$ with $\alpha(x) := \alpha(0)$ for $x \le 0$ and $\alpha(x) := \alpha(1)$ for $x \ge 1$. The condition that $\alpha(x)$ be of bounded variation on $[0,1]$ can be relaxed provided the integral remains defined. This will be done in Example 5 (below) where we consider $\alpha(x) = \log x$ so that $d\alpha(x) = x^{-1}dx$. The above theorem remains true with appropriate changes if the interval [0,1] is replaced by any fixed finite or infinite interval.

(ii) The inner sum in the left-hand side of (1.3) will automatically be convergent when the right-hand side is finite, and likewise in (1.4).

(iii) In Theorem 1 each of the indices m, n can be replaced by a continuous variable. For each such change the corresponding sum must be replaced by an appropriate integral. The following theorem is an example of such a variant of Theorem 1.

Theorem 2. *Let the functions $g(x)$ and $h(x)$ be non-negative and Borel measurable on $[0, \infty)$, and let the functions $f_{t,n}(x)$ $(n = 0, 1, \ldots)$ be non-negative and Borel measurable with respect to x on $[0, \infty)$ for each $t \in [0, \infty)$, and Borel-measurable with respect to t on $[0, \infty)$ for each $x \in [0, \infty)$. Suppose there exist a positive Borel measurable function $p(t)$ on $[0, \infty)$, and positive numbers q_n $(n = 0, 1, \ldots)$ such that*

$$\sum_{n=0}^{\infty} f_{t,n}(x) \le p(t) g(x) \quad \text{for} \quad x \ge 0, \quad t \ge 0,$$

and

$$\int_0^{\infty} f_{t,n}(x)\, dt \le q_n h(x) \quad \text{for} \quad x \ge 0, \quad n = 0, 1, \ldots .$$

Then

$$\sum_{n=0}^{\infty} q_n{}^{1-r} \left| \int_0^{\infty} s(t)\, dt \int_0^{\infty} f_{t,n}(x) d\beta(x) \right|^r$$
$$\le \left(\int_0^{\infty} g(x)^{1/r} h(x)^{1/r'} |d\beta(x)| \right)^r \int_0^{\infty} p(t) |s(t)|^r \, dt ,$$

and

$$\int_0^\infty p(t)^{1-r}\,dt \left|\sum_{n=0}^\infty s_n \int_0^\infty f_{t,n}(x)d\beta(x)\right|^r \le \left(\int_0^\infty g(x)^{1/r'}h(x)^{1/r}|d\beta(x)|\right)^r \sum_{m=0}^\infty q_m|s_m|^r .$$

Proof of Theorem 1. Write

$$s_n(x) := \sum_{m=0}^\infty f_{mn}(x)s_m, \quad t_n := \int_0^1 s_n(x)d\alpha(x) = \sum_{m=0}^\infty s_m \int_0^1 f_{mn}(x)d\alpha(x).$$

By Hölder's inequality and (1), we have

$$\begin{aligned} |s_n(x)| &= \left|\sum_{m=0}^\infty \underbrace{f_{mn}(x)^{1/r}s_m}\cdot\underbrace{f_{mn}(x)^{1/r'}}\right| \\ &\le \left(\sum_{m=0}^\infty f_{mn}(x)|s_m|^r\right)^{1/r}\left(\sum_{m=0}^\infty f_{mn}(x)\right)^{1/r'} \\ &\le (q_n h(x))^{1/r'}\left(\sum_{m=0}^\infty f_{mn}(x)|s_m|^r\right)^{1/r}. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n=0}^\infty q_n^{1-r}|s_n(x)|^r &\le h(x)^{r/r'}\sum_{n=0}^\infty\sum_{m=0}^\infty f_{mn}(x)|s_m|^r \\ &= h(x)^{r/r'}\sum_{m=0}^\infty |s_m|^r \sum_{n=0}^\infty f_{mn}(x) \\ &\le h(x)^{r/r'}g(x)\sum_{m=0}^\infty p_m|s_m|^r. \end{aligned}$$

Therefore, by a Minkowski type inequality (see Hardy et al [7, p. 148]),

$$\begin{aligned} \left(\sum_{n=0}^\infty q_n^{1-r}|t_n|^r\right)^{1/r} &= \left(\sum_{n=0}^\infty q_n^{1-r}\left|\int_0^1 s_n(x)d\alpha(x)\right|^r\right)^{1/r} \\ &\le \int_0^1 \left(\sum_{n=0}^\infty q_n^{1-r}|s_n(x)|^r\right)^{1/r}|d\alpha(x)| \\ &\le \int_0^1 h(x)^{1/r'}f(x)^{1/r}|d\alpha(x)|\left(\sum_{m=0}^\infty p_m|s_m|^r\right)^{1/r}. \end{aligned}$$

This establishes (1.3), and (1.4) follows by an appropriate interchange of symbols. □

2. EXAMPLES INVOLVING GENERALIZED HAUSDORFF TRANSFORMS

Suppose that (λ_n) is a sequence of real numbers with $\lambda_0 \ge 0$, $\lambda_n > 0$ for $n \ge 1$. For $n > m$, let $\lambda_{mn}(t) := 0$, and for $0 \le n \le m$, $0 < t \le 1$, let

$$\lambda_{mn}(t) := -\lambda_{n+1}\cdots\lambda_m \frac{1}{2\pi i}\int_{C_{mn}} \frac{t^z\,dz}{(\lambda_n - z)\cdots(\lambda_m - z)}, \quad \lambda_{mn}(0) := \lambda_{mn}(0+), \quad (2.1)$$

C_{mn} being a positively sensed Jordan contour enclosing $\lambda_n, \dots, \lambda_m$. Here and elsewhere we observe the convention that empty products, like $\lambda_{n+1}\cdots\lambda_m$ when $n = m$, have the value 1. Let $\lambda_{mn} := \int_0^1 \lambda_{mn}(t)\,d\alpha(t)$, and, for $0 \le t \le 1$, $m \ge 1$, let

$$\lambda^*_{mn}(t) := \frac{\lambda_n}{\lambda_m}\lambda_{mn}(t) \quad \text{and} \quad \lambda^*_{mn} := \frac{\lambda_n}{\lambda_m}\lambda_{mn}. \quad (2.2)$$

The triangular matrices (λ_{mn}) and (λ^*_{mn}) are respectively the generalized Hausdorff (see Borwein and Jakimovski [5]) and quasi-Hausdorff matrices. They have been extensively investigated. It is known (see Borwein et al [2]) that $\lambda_{mn}(x) \geq 0$ for $0 \leq x \leq 1,\ 0 \leq n \leq m$, that $\sum_{n=0}^{m} \lambda_{mn}(x) \leq 1$ for $0 \leq x \leq 1,\quad m = 0,1,\ldots,$ and (see Borwein [1]) that $\sum_{m=n}^{\infty} \lambda^*_{mn}(x) \leq 1$ for $0 \leq x \leq 1,\quad n = 1,2,\ldots.$

Example 1a. In Theorem 1 choose

$$f_{mn}(x) = \begin{cases} \lambda_{mn}(x)/\lambda_m & \text{for } m \geq n \geq 1, \\ 0 & \text{otherwise}. \end{cases}$$

Then we have

$$\sum_{n=0}^{\infty} f_{mn}(x) \leq \frac{1}{\lambda_m} \sum_{n=0}^{m} \lambda_{mn}(x) \leq \frac{1}{\lambda_m} \quad \text{for} \quad 0 \leq x \leq 1, \quad m \geq 1,$$

and

$$\sum_{m=0}^{\infty} f_{mn}(x) \leq \frac{1}{\lambda_n} \sum_{m=1}^{\infty} \lambda^*_{mn}(x) \leq \frac{1}{\lambda_n} \quad \text{for} \quad 0 \leq x \leq 1, \quad n \geq 1.$$

Therefore the assumptions of Theorem 1 are satisfied with $p_n = q_n = \frac{1}{\lambda_n}$ for $n \geq 1$, $f(x) = h(x) \equiv 1$. It now follows from (1.4) that

$$\sum_{m=1}^{\infty} \frac{1}{\lambda_m} \left| \sum_{n=1}^{m} \lambda_{mn} s_n \right|^r \leq \left(\int_0^1 |d\alpha(x)| \right)^r \sum_{n=1}^{\infty} \frac{1}{\lambda_n} |s_n|^r . \tag{2.3}$$

This was proved in Borwein [1] under the additional hypotheses

$$0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n \nearrow \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty .$$

It was also proved there that, subject to these hypotheses, the inequality is sharp when $\alpha(x)$ is non-decreasing on $[0,1]$ with $\alpha(0+) = \alpha(0)$.

From (1.3) we get the companion inequality

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left| \sum_{m=n}^{\infty} \lambda^*_{mn} s_m \right|^r \leq \left(\int_0^1 |d\alpha(x)| \right)^r \sum_{n=1}^{\infty} \frac{1}{\lambda_n} |s_n|^r . \tag{2.4}$$

Example 1b. In Theorem 1 choose $f_{mn}(x) = \lambda_{mn}(x)$. Assume that for some $c > 0$ we have

$$\mu := \sup_{0 \leq n \leq m = 0,1,\ldots} \frac{\lambda_{n+1} \ldots \lambda_m}{(\lambda_n + c) \ldots (\lambda_{m-1} + c)} < \infty .$$

Then

$$\sum_{n=0}^{\infty} f_{mn}(x) = \sum_{n=0}^{m} \lambda_{mn}(x) \leq 1 \quad \text{for} \quad 0 \leq x \leq 1.$$

Further, it was proved in Borwein [1] that

$$\sum_{m=0}^{\infty} f_{mn}(x) = \sum_{m=n}^{\infty} \lambda_{mn}(x) \leq \mu x^{-c} \quad \text{for} \quad 0 < x \leq 1.$$

It follows from conclusion (1.3) of Theorem 1 with $p_n = q_n = 1$, $g(x) \equiv 1$, $h(x) = \mu x^{-c}$ that

$$\sum_{n=0}^{\infty} \left| \sum_{m=n}^{\infty} \lambda_{mn} s_m \right|^r \le \mu \left(\int_0^1 x^{-c/r'} |d\alpha(x)| \right)^r \sum_{n=0}^{\infty} |s_n|^r, \tag{2.5}$$

and from (1.4) that

$$\sum_{m=0}^{\infty} \left| \sum_{n=0}^{m} \lambda_{mn} s_n \right|^r \le \mu \left(\int_0^1 x^{-c/r} |d\alpha(x)| \right)^r \sum_{n=0}^{\infty} |s_n|^r. \tag{2.6}$$

Inequality (2.6) shows that the generalized Hausdorff matrix is a bounded operator on l_r with norm at most $\mu^{1/r} \int_0^1 x^{-c/r} |d\alpha(x)|$. This was established by the proof of Theorem 1 in Borwein [1], although the statement of that theorem had the condition $\lambda_{n+1} \le \lambda_n + c$ for $n \ge n_0$, which is strictly stronger than the condition $\mu < \infty$. (See Borwein and Gao [3]). Hardy [6] established (2.6) for ordinary Hausdorff matrices, i.e., $\lambda_n = n$, and showed that in this case the inequality is sharp with $c = \mu = 1$ if $\alpha(x)$ is non-decreasing on $[0,1]$ with $\alpha(0+) = \alpha(0)$.

3. AN INEQUALITY INVOLVING MOMENTS

Example 2. In Theorem 1 choose $f_{mn}(x) = \binom{m+n}{n} x^m (1-x)^n$. We have

$$\sum_{n=0}^{\infty} f_{mn}(x) = x^m \sum_{n=0}^{\infty} \binom{m+n}{n} (1-x)^n = \frac{1}{x} \quad \text{for } 0 < x \le 1\,, m = 0, 1, \dots,$$

and

$$\sum_{m=0}^{\infty} f_{mn}(x) = (1-x)^n \sum_{m=0}^{\infty} \binom{m+n}{m} x^m = \frac{1}{1-x} \quad \text{for } 0 \le x < 1,\ n = 0, 1, \dots.$$

Therefore the assumptions of Theorem 1 are satisfied for $p_n = q_n = 1$; $g(x) = \frac{1}{x}$, $h(x) = \frac{1}{1-x}$. Let $\mu_m := \int_0^1 x^m \, d\alpha(x)$ for $m = 0, 1, \dots$. Then

$$\Delta^n \mu_m = \int_0^1 (1-x)^n x^m \, d\alpha(x) \quad \text{for} \quad m, n = 0, 1, \dots,$$

and hence Theorem 1 yields:

$$\sum_{n=0}^{\infty} \left| \sum_{m=0}^{\infty} s_m \binom{n+m}{n} \Delta^n \mu_m \right|^r \le \left(\int_0^1 x^{-1/r} (1-x)^{-1/r'} |d\alpha(x)| \right)^r \sum_{m=0}^{\infty} |s_m|^r. \tag{3.1}$$

Taking $\alpha(t) = t$ we obtain the following inequality:

$$\sum_{n=0}^{\infty} \left| \sum_{m=0}^{\infty} \frac{s_m}{m+n+1} \right|^r \le \left(\frac{\pi}{\sin \frac{\pi}{r}} \right)^r \left(\sum_{n=0}^{\infty} |s_n|^r \right). \tag{3.2}$$

4. AN INEQUALITY INVOLVING BOREL TRANSFORMS

In this section we suppose that $F(t) := \int_0^\infty e^{-xt} d\beta(x)$ for $t > -\epsilon, 0 < \epsilon$, whence

$$(-1)^n \frac{F^{(n)}(t)}{n!} = \int_0^\infty \frac{x^n e^{-xt}}{n!} d\beta(x) \quad \text{for} \quad t > -\epsilon.$$

Example 3a. Consider the functions $f_{t,n}(x) := \frac{(xt)^n e^{-xt}}{n!}$ for $x \geq 0$, $t \geq 0$, $n = 0, 1, \ldots$. Then

$$\sum_{n=0}^{\infty} f_{t,n}(x) = \sum_{n=0}^{\infty} \frac{(xt)^n e^{-xt}}{n!} = 1 \quad \text{for} \quad x \geq 0, \quad t \geq 0,$$

and

$$\int_0^\infty f_{t,n}(x) dt = \int_0^\infty \frac{(xt)^n e^{-xt}}{n!} dt = \frac{1}{x} \quad \text{for} \quad x > 0, \quad t \geq 0.$$

Hence

$$\int_0^\infty s(t)\, dt \int_0^\infty f_{t,n}(x)\, d\beta(x) = \int_0^\infty t^n s(t)\, dt \int_0^\infty \frac{x^n e^{-xt}}{n!}\, d\beta(x) = \int_0^\infty (-1)^n \frac{t^n F^{(n)}(t)}{n!} s(t)\, dt\,.$$

From Theorem 2 with $p(t) \equiv 1$, $q_n = 1$, $g(x) \equiv 1$, $h(x) = \frac{1}{x}$ we get now

$$\sum_{n=0}^{\infty} \left| \int_0^\infty (-1)^n \frac{t^n F^{(n)}(t)}{n!} s(t) dt \right|^r \leq \left(\int_0^\infty x^{-1/r'} |d\beta(x)| \right)^r \int_0^\infty |s(t)|^r\, dt\,. \tag{4.1}$$

Special cases.

1. If $\beta(x) = 0$ for $0 \leq x < 1$ and $\beta(x) = 1$ for $x \geq 1$, then $F(t) = e^{-t}$ and (4.1) reduces to

$$\sum_{n=0}^{\infty} \left| \int_0^\infty \frac{t^n e^{-t}}{n!} s(t) dt \right|^r \leq \int_0^\infty |s(t)|^r\, dt\,. \tag{4.2}$$

2. If $\beta(x) = 1 - e^{-x}$ for $x \geq 0$, then $F(t) = 1/(t+1)$ and (4.1) reduces to

$$\sum_{n=0}^{\infty} \left| \int_0^\infty \frac{t^n}{(t+1)^{n+1}} s(t) dt \right|^r \leq \left(\Gamma\left(\frac{1}{r}\right) \right)^r \int_0^\infty |s(t)|^r dt\,. \tag{4.3}$$

Example 3b. Consider the functions $f_{m,t}(x) := \frac{(xt)^m e^{-xt}}{m!}$ for $x \geq 0$, $t \geq 0$, $m = 0, 1, \ldots$. Then

$$\sum_{m=0}^{\infty} f_{m,t}(x) = \sum_{m=0}^{\infty} \frac{(xt)^m e^{-xt}}{m!} = 1 \quad \text{for} \quad x \geq 0, \quad t \geq 0,$$

and

$$\int_0^\infty f_{m,t}(x)\, dt = \int_0^\infty \frac{(xt)^m e^{-xt}}{m!} dt = \frac{1}{x} \quad \text{for} \quad x > 0, \quad m = 0, 1, \ldots.$$

Hence

$$\sum_{m=0}^{\infty} s_m \int_0^\infty f_{m,t}(x)\, d\beta(x) = \sum_{m=0}^{\infty} s_m \int_0^\infty \frac{(xt)^m e^{-xt}}{m!}\, d\beta(x) = \sum_{m=0}^{\infty} (-1)^m \frac{t^m F^{(m)}(t)}{m!} s_m\,.$$

From an integral variant of Theorem 1 with $g(x)=\frac{1}{x}$, $\quad h(x)\equiv 1$, we get now

$$\int_0^\infty \left|\sum_{m=0}^\infty (-1)^m \frac{t^m}{m!} F^{(m)}(t) s_m\right|^r dt \le \left(\int_0^\infty x^{-1/r}|d\beta(x)|\right)^r \sum_{m=0}^\infty |s_m|^r . \tag{4.4}$$

This shows that the sequence-to-function $[J, f(x)]$ transformation (see Jakimovski [8]) is a bounded operator from l_r to $L_r[0,\infty)$, with norm at most $\int_0^\infty x^{-1/r}|d\beta(x)|$.

Special cases.

1. If $\beta(x)=0$ for $0\le x<1$ and $\beta(x)=1$ for $x\ge 1$, then $F(t)=e^{-t}$, and (4.4) reduces to

$$\int_0^\infty \left|e^{-t}\sum_{m=0}^\infty \frac{t^m}{m!} s_m\right|^r dt \le \sum_{m=0}^\infty |s_m|^r . \tag{4.5}$$

This means that the sequence-to-function Borel transform is a bounded operator from l_r into $L_r[0,\infty)$, with norm at most 1.

2. If $\beta(x)=1-e^{-x}$ for $x\ge 0$, then $F(t)=1/(t+1)$, and (4.4) reduces to

$$\int_0^\infty \left|\frac{1}{t+1}\sum_{m=0}^\infty s_m\left(\frac{t}{t+1}\right)^m\right|^r dt \le \left(\Gamma\left(\frac{1}{r'}\right)\right)^r \sum_{m=0}^\infty |s_m|^r . \tag{4.6}$$

This shows that the sequence-to-function Abel transform (written in the form $A(t)=\sum_{m=0}^\infty s_m\left(\frac{t}{t+1}\right)^m$) is a bounded operator from l_r into $L_r[0,\infty)$, with norm at most $\Gamma(1/r')$. Making the substitution $x=t/(t+1)$ in the integral we get

$$\int_0^1 (1-x)^{r-2}\left|\sum_{m=0}^\infty s_m x^m\right|^r dx \le \left(\Gamma\left(\frac{1}{r'}\right)\right)^r \sum_{m=0}^\infty |s_m|^r . \tag{4.7}$$

5. INEQUALITIES INVOLVING CONVEX FUNCTIONS

Example 4. Let $w(x)$ be a non-negative, continuous and concave function on $[0,1]$, and let

$$P_{mn}(x) := \binom{m}{n} x^n (1-x)^{m-n} \quad \text{for} \quad m,n=0,1,\dots, \qquad 0\le x\le 1 .$$

In Theorem 1 choose $f_{mn}(x)=P_{mn}(x)w\left(\frac{n}{m}\right)$. We shall show that in this case (1.1) and (1.2) hold with $p_n = q_n = 1$, $\quad g(x) = w(x)$, $\quad h(x) = \frac{w(x)}{x}$. Letting $\mu_n := \int_0^1 t^n d\alpha(t)$, and applying Theorem 1 we will then obtain the following two inequalities:

$$\sum_{n=0}^\infty \left|\sum_{m=n}^\infty \binom{m}{n} \Delta^{m-n}\mu_n w\left(\frac{n}{m}\right) s_m\right|^r \le \left(\int_0^1 w(x)x^{-1/r'}|d\alpha(x)|\right)^r \sum_{m=0}^\infty |s_m|^r , \tag{5.1}$$

and

$$\sum_{m=0}^\infty \left|\sum_{n=0}^m \binom{m}{n} \Delta^{m-n}\mu_n w\left(\frac{n}{m}\right) s_m\right|^r \le \left(\int_0^1 w(x)x^{-1/r}|d\alpha(x)|\right)^r \sum_{m=0}^\infty |s_m|^r . \tag{5.2}$$

In particular, for the function $\alpha(x) = x$ these inequalities become:

$$\sum_{n=0}^{\infty}\left|\sum_{m=n}^{\infty}\frac{1}{m+1}w\left(\frac{n}{m}\right)s_m\right|^r \le \left(\int_0^1 w(x)x^{-1/r'}dx\right)^r \sum_{k=0}^{\infty}|s_k|^r, \tag{5.3}$$

and

$$\sum_{m=0}^{\infty}\left|\frac{1}{m+1}\sum_{n=0}^{m}w\left(\frac{n}{m}\right)s_n\right|^r \le \left(\int_0^1 w(x)x^{-1/r}dx\right)^r \sum_{k=0}^{\infty}|s_k|^r . \tag{5.4}$$

To prove that (1.1) and (1.2) hold we use the following results.

Since the function $w(x)$ is continuous on $[0,1]$ the associated Bernstein polynomial $B_m(x,w) := \sum_{n=0}^{\infty} f_{mn}(x)$ tends uniformly on $[0,1]$ to $w(x)$ as $m \to \infty$. (See Lorentz [12]). Also, since $w(x)$ is concave on [0,1], $B_m(x,w)$ increases with m. (See Temple [15] and Stancu [14]; and Lemma 2 below). Therefore (1.1) holds with $p_n = 1$, $g(x) = w(x)$.

It remains to show that (1.2) holds with $q_n = 1$, $h(x) = w(x)/x$. This will follow from the second conclusion of the following lemma.

The Meyer-König and Zeller approximation operator applied to the function $w(x)$ is defined by $MZ_n(x,w) := \sum_{m=n}^{\infty} xP_{mn}(x)w\left(\frac{n}{m}\right)$, and the divided difference of $w(x)$ on three distinct points $a,b,c \in [0,1]$ is defined to be

$$[a,b,c]_w := \frac{1}{c-a}\left(\frac{w(a)-w(b)}{b-a} - \frac{w(b)-w(c)}{c-b}\right).$$

Lemma 1. (i) *Let $w(x)$ be a bounded function on $[0,1]$. Then*

$$MZ_{n+1}(x,w) - MZ_n(x,w) = -\sum_{m=n+1}^{\infty} xP_{mn}(x)\frac{m-n}{m^2(m+1)}\left[\frac{n}{m},\frac{n+1}{m+1},\frac{n+1}{m}\right]_w$$

for $n = 0,1,\cdots$, $0 \le x \le 1$.

(ii) *If $w(x)$ is non-negative, continuous and concave on $[0,1]$ with $w(0) = 0$, then $MZ_n(x,w)$ increases with n and converges uniformly to $w(x)$ on each interval $[\epsilon,1]$, $0 < \epsilon < 1$, and to $0 = w(0)$ for $x = 0$.*

Proof. (i) It is easy to verify (see Jakimovski [9]) that, for $0 \le n \le m$, $0 \le x \le 1$, we have

$$P_{m+1,n+1}(x) = \frac{m+1}{n+1}P_{mn}(x) - \frac{m-n+1}{n+1}P_{m+1,n}(x).$$

Hence

$$\begin{aligned} MZ_{n+1}(x,w) = &\sum_{m=n+1}^{\infty} xP_{m,n+1}(x)w\left(\tfrac{n+1}{m}\right) = \sum_{m=n}^{\infty} xP_{m+1,n+1}(x)w\left(\tfrac{n+1}{m+1}\right)\\ &= \sum_{m=n}^{\infty}\left\{\tfrac{m+1}{n+1}xP_{mn}(x) - \tfrac{m-n+1}{n+1}xP_{m+1,n}(x)\right\}w\left(\tfrac{n+1}{m+1}\right)\\ &= \sum_{m=n}^{\infty} xP_{mn}(x)\left\{\tfrac{m+1}{n+1}w\left(\tfrac{n+1}{m+1}\right) - \tfrac{m-n}{n+1}w\left(\tfrac{n+1}{m}\right)\right\}, \end{aligned}$$

and so

$$MZ_{n+1}(x,w)-MZ_n(x,w)=$$
$$=\sum_{m=n}^{\infty} xP_{mn}(x)\left\{\tfrac{m+1}{n+1}w\left(\tfrac{n+1}{m+1}\right)-\tfrac{m-n}{n+1}w\left(\tfrac{n+1}{m}\right)-w\left(\tfrac{n}{m}\right)\right\}$$
$$=-\sum_{m=n+1}^{\infty} xP_{mn}(x)\tfrac{m-n}{m^2(m+1)}\left[\tfrac{n}{m},\tfrac{n+1}{m+1},\tfrac{n+1}{m}\right]_w .$$

(ii) Since $w(x)$ is concave on $[0,1]$, we have $\left[\frac{n}{m},\frac{n+1}{m+1},\frac{n+1}{m}\right]_w<0$. Hence, for $0\le x\le 1$, $MZ_n(x,w)$ increases with n. Also, since $w(x)$ is continuous on $[0,1]$, and $w(0)=0$, $MZ_n(x,w)$ converges uniformly on each interval $[\epsilon,1]$, $0<\epsilon<1$, to $w(x)$ and to $0=w(0)$ for $x=0$ (See Meyer-König [13]). □

Example 5. Suppose that $0\le\lambda_0<\lambda_1<\cdots<\lambda_n\nearrow\infty$ and $\sum_{n=1}^{\infty}\frac{1}{\lambda_n}=\infty$. Let $\alpha_{mn}=\left(1-\frac{\lambda_1}{\lambda_{n+1}}\right)\cdots\left(1-\frac{\lambda_1}{\lambda_m}\right)$ for $0\le n<m$, $\alpha_{mm}=1$, and define $\lambda_{mn}(x)$ and $\lambda^*_{mn}(x)$ by (2.1) and (2.2) as before. Given a bounded function $w(x)$ on $[0,1]$ write, for $0\le x\le 1$, $B_m(x,w):=\sum_{n=0}^{m}\lambda_{mn}(x)w(\alpha_{mn})$ for $m=0,1,\cdots$; and

$$QH_n(x,w):=\sum_{m=n}^{\infty}\lambda^*_{mn}w(\alpha_{n-1,m-1})=\sum_{m=n}^{\infty}\frac{\lambda_n}{\lambda_m}\lambda_{mn}(x)w(\alpha_{m-1,n-1})\quad\text{for}\quad n=1,2,\ldots.$$

These are respectively the generalized Bernstein and quasi-Hausdorff approximation operators applied to the function $w(x)$. It is known that if $w(x)$ is continuous on $[0,1]$, then $B_n(x,w)$ converge uniformly on $[0,1]$ to $w(x^{\lambda_1})$ as $n\to\infty$ and $QH_n(x,w)$ converges uniformly on each closed interval in $[\epsilon,1]$, $0<\epsilon<1$, to $w(x)$ and to 0 for $x=0$ (See Jakimovski [8] and Jakimovski and Leviatan [10]).

We will show below that if the function $w(x)$ is non-negative, continuous, non-decreasing and concave on $[0,1]$, then the functions $f_{mn}(x):=\frac{\lambda_{mn}(x)}{\lambda_m}w(\alpha_{m-1,n-1})$, $m,n=1,2,\cdots$, satisfy assumptions (1.1) and (1.2) of Theorem 1 with $p_n=q_n=\frac{1}{\lambda_n}$ for $n\ge1$; $g(x)=h(x)=w(x^{\lambda_1})$. Therefore we will get from Theorem 1 with $\alpha(x)=\log x$ that

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|\sum_{m=n}^{\infty}\frac{s_m}{\lambda_m}w(\alpha_{m-1,n-1})\right|^r\le\left(\frac{1}{\lambda_1}\int_0^1\frac{w(x)}{x}\,dx\right)^r\sum_{m=1}^{\infty}\frac{|s_m|^r}{\lambda_m}. \tag{5.5}$$

We will also show that if the function $w(x)$ is non-negative, continuous, non-increasing and concave on $[0,1]$, then the functions $f_{mn}(x):=\frac{\lambda_{mn}(x)}{\lambda_n}w(\alpha_{mn})$, $m,n=1,2,\ldots$, satisfy assumptions (1.1) and (1.2) of Theorem 1 with $p_n=q_n=\lambda_n$ for $n\ge 1$; $g(x)=h(x)=w(x^{\lambda_1})$. In this case we will get from Theorem 1 with $\alpha(x)=\log x$ the following inequality:

$$\sum_{n=1}^{\infty}\lambda_n\left|\sum_{m=1}^{n}\frac{s_m}{\lambda_m}w(\alpha_{nm})\right|^r\le\left(\frac{1}{\lambda_1}\int_0^1\frac{w(x)}{x}\,dx\right)^r\sum_{m=1}^{\infty}\lambda_m|s_m|^r. \tag{5.6}$$

To prove (5.5) and (5.6) we need the following results.

It is known (see Jakimovski and Russell [11]) that

$$\int_0^1\lambda_{mn}(t)\frac{dt}{t}=\frac{1}{\lambda_n}\quad\text{for}\quad 0<n\le m, \tag{5.7}$$

and that (see Jakimovski [9]), for $0 \le n \le m,\ 0 \le x \le 1$,

$$\lambda_{mn}(x) = \frac{\lambda_{n+1}}{\lambda_{m+1}} \lambda_{m+1,n+1}(x) + \left(1 - \frac{\lambda_n}{\lambda_{m+1}}\right) \lambda_{m+1,n}(x) . \tag{5.8}$$

Hence, for $0 \le x \le 1,\ m \ge 0$, we have

$$\begin{aligned} &B_{m+1}(x,w) - B_m(x,w) \\ &= -\sum_{n=0}^{m-1} \lambda_{m+1,n+1}(x) \left\{ \tfrac{\lambda_{n+1}}{\lambda_{m+1}} w(\alpha_{mn}) - w(\alpha_{m+1,n+1}) + \left(1 - \tfrac{\lambda_{n+1}}{\lambda_{m+1}}\right) w(\alpha_{m,n+1}) \right\} \\ &\quad + \tfrac{\lambda_0}{\lambda_{m+1}} \lambda_{m+1,0}(x) \\ &= -\sum_{n=0}^{m-1} \frac{\alpha^2_{m,n+1}(\lambda_{m+1} - \lambda_{n+1})\lambda_1^2}{\lambda_{n+1}\lambda^2_{m+1}} \lambda_{m+1,n+1}(x) \left[\alpha_{mn}, \alpha_{m+1,n+1}, \alpha_{m,n+1}\right]_w \\ &\quad + \tfrac{\lambda_0}{\lambda_{m+1}} \lambda_{m+1,0}(x). \end{aligned}$$

Since the divided difference $[\alpha_{mn}, \alpha_{m+1,n+1}, \alpha_{m,n+1}]_w$ is negative when $w(x)$ is concave on $[0,1]$, we obtain the following lemma from this identity:

Lemma 2. *If the function $w(x)$ is concave and bounded on $[0,1]$, then $B_m(x,w)$ increases with m on $[0,1]$. If, in addition $w(x)$ is continuous on $[0,1]$, then $B_m(x,w)$ increases with m and converges uniformly to $w(x^{\lambda_1})$ on $[0,1]$.*

Next, by identity (5.8), we have that, for $0 \le x \le 1,\ n \ge 0$,

$$\begin{aligned} &QH_{n+1}(x,w) - QH_n(x,w) \\ &= \sum_{m=n+1}^{\infty} \tfrac{\lambda_{n+1}}{\lambda_m} \lambda_{m,n+1}(x) w(\alpha_{m-1,n}) - \sum_{m=n}^{\infty} \tfrac{\lambda_n}{\lambda_m} \lambda_{mn}(x) w(\alpha_{m-1,n-1}) \\ &= \sum_{m=n+1}^{\infty} \tfrac{\lambda_{n+1}}{\lambda_m} \left\{ \tfrac{\lambda_m}{\lambda_{n+1}} \lambda_{m-1,n}(x) - \tfrac{\lambda_m - \lambda_n}{\lambda_{n+1}} \lambda_{mn}(x) \right\} w(\alpha_{m-1,n}) \\ &\quad - \sum_{m=n}^{\infty} \tfrac{\lambda_n}{\lambda_m} \lambda_{mn}(x) w(\alpha_{m-1,n-1}) \\ &= \sum_{m=n+1}^{\infty} \lambda_{mn}(x) \left\{ w(\alpha_{mn}) - \tfrac{\lambda_m - \lambda_n}{\lambda_m} w(\alpha_{m-1,n}) - \tfrac{\lambda_n}{\lambda_m} w(\alpha^{\lambda_1}_{m-1,n-1}) \right\} \\ &= -\sum_{m=n+1}^{\infty} \frac{\lambda_1^2(\lambda_m - \lambda_n)\alpha^2_{m-1,n}}{\lambda_n \lambda_m^2} \lambda_{mn}(x) \left[\alpha_{m-1,n-1}, \alpha_{mn}, \alpha_{m-1,n}\right]_w . \end{aligned}$$

From this identity we get the following lemma:

Lemma 3. *If the function $w(x)$ is concave and bounded on $[0,1]$, then $QH_n(x,w)$ increases with n on $[0,1]$. If, in addition $w(x)$ is continuous on $[0,1]$, then $QH_n(x,w)$ increases with n and converges uniformly to $w(x^{\lambda_1})$ on $[0,1]$.*

Proof of inequality (5.5). Assume $w(x)$ is continuous, concave and increasing on [0,1]. Then we have, for $f_{mn}(x) = \frac{\lambda_{mn}(x)}{\lambda_m} w(\alpha_{m-1,n-1})$, that

$$\begin{aligned} \lambda_m \sum_{n=1}^{\infty} f_{mn}(x) &= \sum_{n=1}^{m} \lambda_{mn}(x) w(\alpha_{m-1,n-1}) \le \sum_{n=1}^{m} \lambda_{mn}(x) w(\alpha_{mn}) \\ &\le \sum_{n=0}^{m} \lambda_{mn}(x) w(\alpha_{mn}) \le w(x^{\lambda_1}), \end{aligned}$$

by Lemma 1. Also

$$\lambda_n \sum_{m=1}^{\infty} f_{mn}(x) = \sum_{m=n}^{\infty} \lambda^*_{mn}(x) w(\alpha_{m-1,n-1}) \le w(x^{\lambda_1}),$$

by Lemma 2. Further, for $\alpha(x) = \log x$,

$$\int_0^1 f_{mn}(x) d\alpha(x) s_m = \frac{\lambda_n}{\lambda_m} w(\alpha_{m-1,n-1}) s_m \int_0^1 \frac{\lambda_{mn}(x)}{x} dx = \frac{s_m}{\lambda_m} w(\alpha_{m-1,n-1}).$$

It follows, by Theorem 1 with $p_n = q_n = 1/\lambda_n$, $g(x) = h(x) = w(x^{\lambda_1})$, $\alpha(x) = \log x$, that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left| \sum_{m=n}^{\infty} \frac{s_m}{\lambda_m} w(\alpha_{m-1,n-1}) \right|^r &\le \left(\int_0^1 \frac{w(x^{\lambda_1})}{x} dx \right)^r \sum_{m=1}^{\infty} \frac{|s_m|^r}{\lambda_m} \\ &= \left(\frac{1}{\lambda_1} \int_0^1 \frac{w(x)}{x} dx \right)^r \sum_{m=1}^{\infty} \frac{|s_m|^r}{\lambda_m}.\end{aligned}$$

□

Proof of inequality (5.6). Assume $w(x)$ is bounded, non-increasing and concave on $[0,1]$. Then we have, for

$$f_{mn}(x) = \lambda_n \lambda_{nm}(x) w(\alpha_{nm}), \quad n \ge m \ge 0,$$

that

$$\begin{aligned}\frac{1}{\lambda_m} \sum_{n=0}^{\infty} f_{mn}(x) &= \sum_{n=m}^{\infty} \frac{\lambda_n}{\lambda_m} \lambda_{nm}(x) w(\alpha_{nm}) = \sum_{n=m}^{\infty} \lambda^*_{nm}(x) w(\alpha_{nm}) \\ &\le \sum_{n=m}^{\infty} \lambda^*_{nm}(x) w(\alpha_{n-1,m-1}) \le w(x^{\lambda_1}),\end{aligned}$$

by Lemma 2. Now $\frac{1}{\lambda_n} \sum_{m=0}^{\infty} f_{mn}(x) = \sum_{m=0}^{n} \lambda_{nm}(x) w(\alpha_{nm}) \le w(x^{\lambda_1})$, by Lemma 1. Further, for $\alpha(x) = \log x$,

$$\int_0^1 f_{mn}(x) d\alpha(x) s_m = \lambda_n w(\alpha_{nm}) s_m \int_0^1 \frac{\lambda_{nm}(x)}{x} dx = \frac{\lambda_n}{\lambda_m} s_m w(\alpha_{nm}).$$

It follows, by Theorem 1 with $p_n = q_n = \lambda_n$, $g(x) = h(x) = w(x^{\lambda_1})$, $\alpha(x) = \log x$ that

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_n \left| \sum_{m=1}^{n} \frac{s_m}{\lambda_m} s_m w(\alpha_{nm}) \right|^r &\le \left(\int_0^1 \frac{w(x^{\lambda_1})}{x} dx \right)^r \sum_{m=1}^{\infty} \lambda_m |s_m|^r \\ &= \left(\frac{1}{\lambda_1} \int_0^1 \frac{w(x)}{x} dx \right)^r \sum_{m=1}^{\infty} \lambda_m |s_m|^r.\end{aligned}$$

□

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REFERENCES

[1] D. Borwein, Generalized Hausdorff matrices as bounded operators on l_p, *Math. Z.,* **83** (1983), 483–487.

[2] D. Borwein, F.P. Cass and J.E. Sayre, On generalized Hausdorff matrices, *Arch. Math.,* **46** (1986), 354–360.

[3] D. Borwein and X. Gao, Generalized Hausdorff and weighted mean matrices as operators on l_p, *J. Math. Anal. Appl.,* **78** (1993), 517–528.

[4] D. Borwein and A. Jakimovski, Matrix operators on l_p, *Rocky Mountain J. Math.,* **9** (1979), 463–477.

[5] D. Borwein and A. Jakimovski, Generalization of the Hausdorff moment problem, *Canad. J. Math.,* **33** (1981), 946–960.

[6] G.H. Hardy, An inequality for Hausdorff means, *J. London Math. Soc.,* **18** (1943), 46–50.

[7] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities,* 2nd edn., Cambridge University Press, Cambridge, 1952.

[8] A. Jakimovski, The sequence to function transformations analogous to Hausdorff transformations, *Bull. of the Research Council of Israel,* **18F** (1960), 135–154.

[9] A. Jakimovski, Generalized Bernstein polynomials for discontinuous and convex functions, *J. d'Analyse Math.,* **23** (1970), 171–183.

[10] A. Jakimovski and D. Leviatan, A property of approximation operators and applications to Tauberian constants, *Math. Zeit.,* **102** (1967), 177–204.

[11] A. Jakimovski and D.C. Russell, On the Hausdorff moment problem, Proc. of the First "International Conference on General Inequalities", Oberwolfach (1976), 63-81. Edited by E.F. Beckenbach, Birckenbach Verlag, Basel, Stuttgart, 1977.

[12] G.G. Lorentz, *Bernstein Polynomials,* University of Toronto Press, Toronto, 1953.

[13] W. Meyer-König, Bernsteinsche Potenzreihen, *Studia Math.,* **19** (1960), 89–94.

[14] D.D. Stancu, On the monotonicity of the sequence formed by the first order derivatives of the Bernstein polynomials , *Math. Z.,* **98** (1967), 46–51.

[15] W.B. Temple, Stieltjes integral representation of convex functions, *Duke Math. J.,* **21** (1954), 527–531.