SIMPLE CONDITIONS FOR MATRICES TO BE BOUNDED OPERATORS ON l_p

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ABSTRACT. The two theorems proved yield simple yet reasonably general conditions for triangular matrices to be bounded operators on l_p. The theorems are applied to Nörlund and weighted mean matrices.

1. **Introduction.** Suppose throughout that

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and that $A := (a_{nk})_{n,k \geq 0}$ is a triangular matrix of non-negative real numbers, that is $a_{nk} \geq 0$ for $n, k \geq 0$, and $a_{nk} = 0$ for $n > k$. Let l_p be the Banach space of all complex sequences $x = (x_n)_{n \geq 0}$ with norm

$$\|x\|_p := \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} < \infty,$$

and let $B(l_p)$ be the Banach algebra of all bounded linear operators on l_p. Thus $A \in B(l_p)$ if and only if $Ax \in l_p$ whenever $x \in l_p$, Ax being the sequence with n-th term $(Ax)_n := \sum_{k=0}^{n} a_{nk}x_k$. Let

$$\|A\|_p := \sup_{\|x\|_p \leq 1} \|Ax\|_p,$$

so that $A \in B(l_p)$ if and only if $\|A\|_p < \infty$, in which case $\|A\|_p$ is the norm of A.

We shall prove the following two theorems:

THEOREM 1. *Suppose that*

(1)
$$M_1 := \sup_{n \geq 0} \sum_{k=0}^{n} a_{nk} < \infty,$$

(2)
$$M_2 := \sup_{\substack{0 \leq k \leq n/2 \\ n \geq 0}} (n+1)a_{nk} < \infty,$$

and

(3)
$$M_3 := \sup_{k \geq 0} \sum_{n=k}^{2k} a_{nk} < \infty.$$

Then $A \in B(l_p)$ and $\|A\|_p \leq \mu_1^{1/q}\mu_2^{1/p}$, where

$$\mu_1 \leq 2^{1/p}M_1 + qM_2 \quad and \quad \mu_2 \leq M_3 + qM_2.$$

THEOREM 2. *Suppose that (1) holds, and that*

(4)
$$a_{nk} \leq M_4 a_{nj} \quad for \ 0 \leq k \leq j \leq n,$$

where M_4 is a positive number independent of k, j, n. Then $A \in B(l_p)$ and

$$\max\left(a_{00}, \frac{\lambda q}{M_4} \right) \leq \|A\|_p \leq qM_1 M_4^{q-1},$$

where $\lambda := \liminf n a_{n0}$.

These theorems yield simple yet fairly general conditions for $A \in B(l_p)$. In Section 4 we shall illustrate their scope by applying them to Nörlund and weighted mean matrices.

2. **Lemmas.** We require the following known results:

LEMMA 1 (SEE [4, THEOREM 2]). *If*

$$\mu_1 := \sup_{n \geq 0} \sum_{k=0}^{n} a_{nk}\left(\frac{n+1}{k+1} \right)^{1/p} < \infty \quad and \quad \mu_2 := \sup_{k \geq 0} \sum_{n=k}^{\infty} a_{nk}\left(\frac{k+1}{n+1} \right)^{1/q} < \infty,$$

then $A \in B(l_p)$ and $\|A\|_p \leq \mu_1^{1/q}\mu_2^{1/p}$.

LEMMA 2 (SEE [10, LEMMA 4] AND [8, LEMMA 1]). *If $q > 1$ and $z_n \geq 0$ for $n = k, k+1, \ldots,$ where k is a non-negative integer, then*

$$\left(\sum_{n=k}^{\infty} z_n \right)^q \leq q \sum_{n=k}^{\infty} z_n \left(\sum_{j=n}^{\infty} z_j \right)^{q-1}.$$

3. **Proofs of the theorems.**

PROOF OF THEOREM 1. Let $s := 1/p$, $t := 1/q$, and let μ_1, μ_2 be as in Lemma 1. Then, by (2),

$$(n+1)^s \sum_{0 \leq k \leq n/2} \frac{a_{nk}}{(k+1)^s} \leq \left(\sup_{0 \leq k \leq n/2} a_{nk} \right)(n+1)^s \sum_{0 \leq k \leq n/2} \frac{1}{(k+1)^s}$$

$$\leq \left(\sup_{0 \leq k \leq n/2} a_{nk} \right) \frac{(n+1)^s(n+2)^{1-s}}{(1-s)2^{1-s}} \leq \frac{M_2}{1-s} = qM_2;$$

and, by (1),

$$(n+1)^s \sum_{n/2 < k \leq n} \frac{a_{nk}}{(k+1)^s} \leq \frac{(n+1)^s 2^s}{(n+2)^s}M_1 \leq 2^s M_1.$$

Hence

$$\mu_1 \le 2^s M_1 + q M_2.$$

Also, by (2),

$$(k+1)^t \sum_{n=2k+1}^{\infty} \frac{a_{nk}}{(n+1)^t} \le M_2 (k+1)^t \sum_{n=2k+1}^{\infty} \frac{1}{(n+1)^{t+1}}$$

$$\le M_2 (k+1)^t \int_{2k}^{\infty} \frac{dx}{(x+1)^{t+1}} = M_2 \frac{q(k+1)^t}{(2k+1)^t} \le q M_2;$$

and, by (3),

$$(k+1)^t \sum_{n=k}^{2k} \frac{a_{nk}}{(n+1)^t} \le M_3.$$

Hence

$$\mu_2 \le q M_2 + M_3.$$

The desired conclusion now follows from Lemma 1. ∎

PROOF OF THEOREM 2. Our proof is modelled on the proof given by Johnson, Mohapatra and Ross of Theorem 1 in [9]. Let T be the transpose of A. We shall use the familiar result that $A \in B(l_p)$ if and only if $T \in B(l_q)$ and $\|A\|_p = \|T\|_q$. Let $y = Tx$ where $x = (x_n)$ is a real non-negative sequence in l_q. Then, by Lemma 2, (4), and Hölder's inequality,

$$\|y\|_q^q = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_{nk} x_n \right)^q \le q \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_{nk} x_n \left(\sum_{j=n}^{\infty} a_{jk} x_j \right)^{q-1}$$

$$\le q M_4^{q-1} \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_{nk} x_n \left(\sum_{j=n}^{\infty} a_{jn} x_j \right)^{q-1} = q M_4^{q-1} \sum_{n=0}^{\infty} x_n y_n^{q-1} \sum_{k=0}^{n} a_{nk}$$

$$\le q M_1 M_4^{q-1} \sum_{n=0}^{\infty} x_n y_n^{q-1} \le q M_1 M_4^{q-1} \left(\sum_{n=0}^{\infty} x_n^q \right)^{1/q} \left(\sum_{n=0}^{\infty} y_n^q \right)^{1/p}$$

$$= q M_1 M_4^{q-1} \|x\|_q \|y\|_q^{q/p}.$$

It follows that $\|y\|_q \le q M_1 M_4^{q-1} \|x\|_q$, and hence that $\|T\|_q = \|A\|_p \le q M_1 M_4^{q-1}$.

To establish the lower estimate for $\|A\|_p$, fix $\delta \in (0, 1)$ and choose a positive integer N so large that $n a_{n0} > \delta \lambda$ for all $n \ge N$. Suppose $M > N$ and define $x = (x_n)$ by setting $x_n := n^{-1/p}$ for $N \le n \le M$, and $x_n := 0$ otherwise. Then, by (4),

$$\|Ax\|_p^p \ge \sum_{n=N}^{M} \left(\sum_{k=N}^{n} a_{nk} x_k \right)^p \ge \left(\frac{\delta \lambda}{M_4} \right)^p \sum_{n=N}^{M} \left(\frac{1}{n} \sum_{k=N}^{n} k^{-1/p} \right)^p$$

$$\ge \left(\frac{\delta \lambda}{M_4} \right)^p \sum_{n=N}^{M} \left(\frac{1}{n} \int_N^n x^{-1/p} dx \right)^p = \left(\frac{\delta \lambda q}{M_4} \right)^p \sum_{n=N}^{M} \frac{1}{n} \left(1 - \left(\frac{N}{n} \right)^{1/q} \right)^p$$

$$= \left(\frac{\delta \lambda q}{M_4} \right)^p \rho_M \sum_{n=N}^{M} \frac{1}{n} = \left(\frac{\delta \lambda q}{M_4} \right)^p \rho_M \|x\|_p^p,$$

where $\rho_M \to 1$ as $M \to \infty$. It follows that $\|A\|_p \ge \frac{\delta \lambda q}{M_4}$ and hence, since δ can be arbitrarily close to 1 in $(0, 1)$, that $\|A\|_p \ge \frac{\lambda q}{M_4}$. Finally, for the unit coordinate sequence $e_0 = (1, 0, 0, \ldots)$, we have $\|Ae_0\|_p \ge a_{00} \|e_0\|_p$, so that $\|A\|_p \ge a_{00}$. ∎

4. **Examples involving Nörlund and weighted mean matrices.** Let $a := (a_n)$ be a sequence of real non-negative numbers with $a_0 > 0$, and let $A_n := a_0 + a_1 + \cdots + a_n$.

The *Nörlund matrix* N_a is defined to be the triangular matrix (a_{nk}) with $a_{nk} := \frac{a_{n-k}}{A_n}$ for $0 \le k \le n$, and $a_{nk} := 0$ for $k > n$.

The *weighted mean matrix* M_a is defined to be the triangular matrix (a_{nk}) with $a_{nk} := \frac{a_k}{A_n}$ for $0 \le k \le n$, and $a_{nk} := 0$ for $k > n$.

Observe that

$$\sum_{k=0}^{n} \frac{a_{n-k}}{A_n} = 1 \quad \text{and} \quad \sum_{n=k}^{2k} \frac{a_{n-k}}{A_n} \le \frac{1}{A_k} \sum_{n=k}^{2k} a_{n-k} = 1,$$

so that the Nörlund matrix N_a automatically satisfies conditions (1) and (3) of Theorem 1 with $M_1 = 1$ and $M_3 \le 1$. The weighted mean matrix M_a also satisfies (1) with $M_1 = 1$.

EXAMPLE 1. Suppose that

$$(5) \qquad M_2' := \sup_{n \ge 0} \frac{(n+1)a_n}{A_n} < \infty.$$

It is immediate that, for the Nörlund matrix N_a, (2) implies (5) with $M_2' \le M_2$. On the other hand we have, for $0 \le k \le n/2$,

$$\frac{(n+1)a_{n-k}}{A_n} = \frac{(n+1-k)a_{n-k}}{A_{n-k}} \cdot \frac{A_{n-k}}{A_n} \cdot \frac{n+1}{n+1-k} \le 2 \frac{(n+1-k)a_{n-k}}{A_{n-k}} \le 2 M_2',$$

so that (5) implies (2) with $M_2 \le 2 M_2'$ for the Nörlund matrix N_a.

It follows now from Theorem 1 that, subject to (5), $N_a \in B(l_p)$ and $\|N_a\|_p \le \mu_1^{1/q} \mu_2^{1/p}$, where

$$\mu_1 \le 2^{1/p} + 2 q M_2' \quad \text{and} \quad \mu_2 \le 1 + 2 q M_2'.$$

This result was proved directly by Borwein and Cass in [3] with a slightly different and better estimate for the upper bound of the operator norm. See also [2] and [7] for related results.

EXAMPLE 2. Suppose that (a_n) is *non-increasing*. It is immediate that this implies (5) with $M_2' \le 1$, but it also implies (4) with $M_4 = 1$ for the Nörlund matrix N_a. Hence either Theorem 1 or Theorem 2 yields that $N_a \in B(l_p)$, and Theorem 2 shows that

$$\max(1, \lambda q) \le \|N_a\|_p \le q,$$

where $\lambda := \liminf \frac{n a_n}{A_n}$. This result was proved as Theorem 1 by Johnson, Mohapatra and Ross in [9]. Our Theorem 2 is clearly a generalization of their theorem.

EXAMPLE 3.   Suppose that (a_n) *is non-decreasing.* Evidently the weighted mean matrix M_a satisfies (4) with $M_4 = 1$. It follows from Theorem 2 that $M_a \in B(l_p)$ with $\|M_a\|_p \le q$. This result was first proved by Cartlidge [6] by an entirely different method. See also [1], [5] and [7] for related and more general results.

The preceding examples involved proofs of known results. For the next example we use Theorem 2 to prove a new result which combines Examples 2 and 3. Let $a := (a_n)$, $b := (b_n)$ be sequences of real non-negative numbers with $a_0 > 0$, $b_0 > 0$, and let $c_n := a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$. The *generalized Nörlund matrix* $N_{a,b}$ is defined to be the triangular matrix (a_{nk}) with $a_{nk} := \frac{a_{n-k} b_k}{c_n}$ for $0 \le k \le n$, and $a_{nk} := 0$ for $k > n$.

EXAMPLE 4.   Suppose (a_n) is non-decreasing and (b_n) is non-increasing. Then $N_{a,b} \in B(l_p)$ and $\max(1, \lambda q) \le \|N_{a,b}\|_p \le q$, where $\lambda := \liminf \frac{n a_n b_0}{c_n}$.

PROOF.   Evidently the matrix $N_{a,b}$ satisfies (1) and (4) with $M_1 = M_4 = 1$. The desired conclusions follow from Theorem 2.   ∎

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