MATRIX TRANSFORMATIONS OF SERIES OF ORTHOGONAL POLYNOMIALS

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Abstract. For a sequence of polynomials (P_n) orthonormal on the interval $[-1, 1]$, we consider the sequence of transforms (g_n) of the series $\sum_{k=0}^{\infty} a_k P_k(u)$ given by $g_n(u) = \sum_{k=0}^{\infty} b_{nk} a_k P_k(u)$. We establish necessary and sufficient conditions on the matrix (b_{nk}) for the sequence (g_n) to converge uniformly on compact subsets of the interior of an appropriate ellipse to a function holomorphic on that interior.

§1. *Introduction.* Suppose throughout that $1 < P \leqslant \infty$, $1 < R < \infty$, and that all sequences and matrices are complex with indices running through $0, 1, 2, \ldots$. We make the following definitions.

$\mathbb{C}$ is the finite complex plane.

γ_R is the ellipse with foci ± 1 and half-axes $a = \frac{1}{2}(R + R^{-1})$, $b = \frac{1}{2}(R - R^{-1})$. Note that an ellipse with foci ± 1 having R as the sum of its two half-axes is necessarily γ_R,

D_R^{γ} is the interior of the ellipse γ_R, and $D_{\infty}^{\gamma} = \mathbb{C}$,

(P_n) is an orthonormal sequence of polynomials with respect to a fixed non-negative weight function w on the interval $[-1, 1]$. That is, P_n is a polynomial of degree n, and

$$\int_{-1}^{1} P_n(u) P_m(u) w(u) du = \delta_{nm}.$$

We assume throughout that

$$w \in L(-1, 1) \quad \text{and} \quad w^{-\varepsilon} \in L(-1, 1) \quad \text{for some} \quad \varepsilon > 0.$$

The first of these integrability conditions is standard, and the second is imposed for the purposes of the present paper. The classical Jacobi polynomials, for which $w(u) = (u-1)^{\alpha}(u+1)^{\beta}$ with $\alpha, \beta > -1$, satisfy the conditions.

$\mathscr{E}$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\lim |a_n|^{1/(n+1)} = 0$.

$\mathscr{E}^{\beta}$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\limsup |a_n|^{1/(n+1)} < \infty$.

$\mathscr{E}_R$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\sum_{n=0}^{\infty} |a_n| R^n < \infty$.

$\mathbf{A}_R$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\limsup |a_n|^{1/(n+1)} = 1/R$.

The following lemma, the proof of which appears in **[1]**, shows that $\mathscr{E}^{\beta}$ is the β-dual of $\mathscr{E}$.

LEMMA 1. *A sequence* $\mathbf{b}$ *has the property that* $\sum_{n=0}^{\infty} b_n a_n$ *is convergent for each* $\mathbf{a} \in \mathscr{E}$ *if, and only if,* $\mathbf{b} \in \mathscr{E}^{\beta}$.

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The following are the first three of eight theorems we shall prove concerning matrix transformations of series of orthogonal polynomials. They are analogues of Theorems 1, 2 and 3 in [1] concerning matrix transformations of power series.

THEOREM 1. *A matrix* $\mathbf{B}\equiv(b_{nk})$ *has the property that whenever the sequence* $\mathbf{a}\equiv(a_n)\in\mathscr{E}_R$ *the sequence of functions* (g_n) *given by*

$$g_n(u)=\sum_{k=0}^{\infty} b_{nk}a_kP_k(u), \qquad n=0, 1, \ldots,$$

converges uniformly on every compact subset of D_P^γ, *each series* $\sum_{k=0}^{\infty} b_{nk}a_kP_k(u)$ *of orthogonal polynomials being convergent on* D_P^γ, *if, and only if,*

(i) $\lim_{n\to\infty} b_{nk}=b_k$ *for* $k=0, 1, \ldots$; *and*

(ii) $M(p)=\sup_{n\geq 0,k\geq 0}|b_{nk}|(p/R)^k<\infty$ *whenever* $1<p<P$.

And then $\lim_{n\to\infty} g_n(u)=\sum_{k=0}^{\infty} b_ka_kP_k(u)$ *on* D_P^γ.

THEOREM 2. *A matrix* $\mathbf{B}\equiv(b_{nk})$ *has the property that whenever the sequence* $\mathbf{a}\equiv(a_n)\in\mathbf{A}_R$ *the sequence of functions* (g_n) *given by*

$$g_n(u)=\sum_{k=0}^{\infty} b_{nk}a_kP_k(u), \qquad n=0, 1, \ldots,$$

converges uniformly on every compact subset of D_P^γ, *each series* $\sum_{k=0}^{\infty} b_{nk}a_{jk}P_k(u)$ *of orthogonal polynomials being convergent on* D_P^γ, *if, and only if,*

(i) $\lim_{n\to\infty} b_{nk}=b_k$ *for* $k=0, 1, \ldots$; *and*

(ii) $M(p)=\sup_{n\geq 0,k\geq 0}|b_{nk}|(p/R)^k<\infty$ *whenever* $1<p<P$.

And then $\lim_{n\to\infty} g_n(u)=\sum_{k=0}^{\infty} b_ka_kP_k(u)$ *on* D_P^γ.

THEOREM 3. *A matrix* $\mathbf{B}\equiv(b_{nk})$ *has the property that whenever the sequence* $\mathbf{a}\equiv(a_n)\in\mathscr{E}$ *the sequence of functions* (g_n) *given by*

$$g_n(u)=\sum_{k=0}^{\infty} b_{nk}a_kP_k(u), \qquad n=0, 1, \ldots,$$

converges uniformly on every compact subset of $\mathbb{C}$, *each series* $\sum_{k=0}^{\infty} b_{nk}a_kP_k(u)$ *of orthogonal polynomials being convergent on* $\mathbb{C}$, *if, and only if,*

(i) $\lim_{n\to\infty} b_{nk}=b_k$ *for* $k=0, 1, \ldots$;

(ii) $M=\sup_{n\geq 0,k\geq 0}|b_{nk}|^{1/(k+1)}<\infty$.

And then $\lim_{n\to\infty} g_n(u)=\sum_{k=0}^{\infty} b_ka_kP_k(u)$ *on* $\mathbb{C}$.

These theorems show that if the series-to-sequence transform given by $\mathbf{B}$ is regular, then it is necessary in each case that $\lim_{n\to\infty} b_{nk}=b_k=1$ for $k=0, 1, \ldots$, and this in turn implies that $P\leq R$ in Theorems 1 and 2 (*i.e.*, the sequence (g_n) cannot converge uniformly in the interior of any ellipse γ_P with $P>R$). Regular sequence-to-sequence transforms of power series have been considered by Peyerimhoff [8] and Luh [7] among others. One of the novel features of our approach is that we deal with series-to-sequence transforms rather than sequence-to-sequence transforms.

Let (B_n) be a sequence of non-zero complex numbers. The associated Nörlund series-to-sequence matrix $\mathbf{N}_B$ is the triangular matrix (b_{nk}) with

$$b_{nk}=\begin{cases}\dfrac{B_{n-k}}{B_n}, & \text{if} \quad 0\leq k\leq n,\\ 0, & \text{otherwise.}\end{cases}$$

The following theorem is an immediate consequence of Theorem 1.

THEOREM N. *The Nörlund matrix* $\mathbf{N}_B$ *has the property that whenever the sequence* $\mathbf{a}\equiv(a_n)\in\mathscr{E}_R$ *the sequence of functions* (g_n) *given by*

$$g_n(u)=\frac{1}{B_n}\sum_{k=0}^{n} B_{n-k}a_kP_k(u), \qquad n=0, 1, \ldots,$$

converges uniformly on every compact subset of D_P^γ, *if, and only if,*

$$\lim_{n\to\infty}\frac{B_{n-1}}{B_n}=b \qquad \text{with} \qquad |b|=\frac{R}{P}.$$

And then $\lim_{n\to\infty} g_n(u)=\sum_{k=0}^{\infty} b^ka_kP_k(u)$ *on* D_P^γ.

Note. In view of Theorem 2, Theorem N remains true if $\mathscr{E}_R$ is replaced by $\mathbf{A}_R$.

§2. *Orthogonal polynomials.* In this section we set out some of the properties of orthogonal polynomials required in our proofs. Note that the function $u=\frac{1}{2}(z+z^{-1})$ maps the region $\{z:|z|>1\}$ bijectively onto the region $\{u: u\notin[-1, 1]\}$, and that each circle $|z|=R$ is mapped onto γ_R. The inverse of this function is $z=u+\sqrt{u^2-1}$. Here and elsewhere in the paper the sign of the square root is chosen so that $|u+\sqrt{u^2-1}|>1$ when $u\notin[-1, 1]$. We then have, for $z=u+\sqrt{u^2-1}$, that $|z|=R$ when $u\in\gamma_R$, and $|z|<R$ when $u\in D_R^\gamma$. The function $u=\frac{1}{2}(z+z^{-1})$ maps both the top half and the bottom half of the unit circle $\{z:|z|=1\}$ onto $[-1, 1]$.

LEMMA 2. *For* $\varepsilon>0$ *let the non-negative weight function* $w\in L(-1, 1)$ *associated with the orthonormal sequence of polynomials* (P_n) *be such that* $w^{-\varepsilon}\in L(-1, 1)$, *and let* $|z|\geq 1$ *and* $u=\frac{1}{2}(z+z^{-1})$. *Then*

$$|P_n(u)|\leq K(\varepsilon)(1+n)^{2+(2/\varepsilon)}|z|^n \qquad \text{for} \qquad n=0, 1, \ldots,$$

where $K(\varepsilon)$ *is a positive number independent of* n.

Proof. By Bernstein's inequality (see [5, Theorem 7])

$$|P_n(u)| \leqslant \max_{-1 \leqslant t \leqslant 1} |P_n(t)||z|^n,$$

and by a result due to Erdéli [2, Theorem 5]

$$\max_{-1 \leqslant t \leqslant 1} |P_n(t)| \leqslant K_1(\varepsilon)(1+n)^{2+(2/\varepsilon)} \int_{-1}^{1} |P_n(t)| w(t) dt.$$

Finally, by the Cauchy–Schwarz inequality,

$$\int_{-1}^{1} |P_n(t)| w(t) dt \leqslant \left(\int_{-1}^{1} P_n(t)^2 w(t) dt \right)^{1/2} \left(\int_{-1}^{1} w(t) dt \right)^{1/2} = \left(\int_{-1}^{1} w(t) dt \right)^{1/2}.$$

Combining the above inequalities we get the required result.

LEMMA 3. (Expansion of a holomorphic function in terms of orthogonal polynomials). *Let the non-negative weight function* $w \in L(-1, 1)$ *associated with the orthonormal sequence of polynomials* (P_n) *be such that* $w^{-\varepsilon} \in L(-1, 1)$ *for some* $\varepsilon > 0$. *Let* $f(u)$ *be holomorphic on the closed segment* $[-1, 1]$, *and let* γ_R *denote the largest ellipse with foci* ± 1 *on the interior of which* $f(u)$ *is holomorphic. The Fourier series expansion of* $f(u)$ *on* D_R^γ, *the interior of* γ_R, *is given by*

$$f(u) = \sum_{k=0}^{\infty} a_k P_k(u),$$

where

$$a_k = \int_{-1}^{1} f(t) P_k(t) w(t) dt.$$

The Fourier series is absolutely convergent on D_R^γ, *and is also uniformly convergent on compact subsets of* D_R^γ. *It is divergent on the exterior of* γ_R. *Further, the sum* R *of the semi-axes of the ellipse of convergence is given by*

$$\frac{1}{R} = \limsup_{k \to \infty} |a_k|^{1/k}.$$

Proof. All but the statement about absolute convergence follows from Theorems 12.7.3 and 12.7.4 in [11], since the conditions on the weight w are more stringent than those in the said theorems. To prove the absolute convergence part, let

$$\frac{1}{R} = \limsup_{k \to \infty} |a_k|^{1/k},$$

and let $u \in D_R^\gamma$. Then $R > 1$ and $u = \frac{1}{2}(z + z^{-1})$ with $1 \leqslant |z| < R$. Let $|z| < R_0 < R$. Then $|a_k| < R_0^{-k}$ for all sufficiently large k. Hence, by Lemma 2,

$$|a_k P_k(u)| = (|a_k||z|^k)|z^{-k} P_k(u)| \leqslant K(\varepsilon)(1+k)^{2+(2/\varepsilon)} \left(\frac{|z|}{R_0} \right)^k$$

for all sufficiently large k, and therefore $\sum_{k=0}^{\infty} |a_k P_k(u)|$ is convergent.

LEMMA 4. (Cauchy-type inequalities for Fourier series). *Let the non-negative weight function* $w \in L(-1, 1)$ *associated with the orthonormal sequence of polynomials* (P_n) *be such that* $w^{-\varepsilon} \in L(-1, 1)$ *for some* $\varepsilon > 0$. *Assume that the function* $f(u)$ *is holomorphic on* D_R^γ *and continuous on* $\bar{D}_R^\gamma$, *the closure of* D_R^γ. *Let* $\sum_{k=0}^{\infty} a_k P_k(u)$ *be its Fourier series. Then*

$$|a_n| \leqslant \frac{c(R)}{R^n} \max_{u \in \gamma_R} |f(u)| \qquad \text{for} \qquad n = 0, 1, \ldots,$$

where $c(R) = (2R/R-1) \left(\int_{-1}^{1} w(t) dt\right)^{1/2}$.

Proof. Suppose first that $n \geqslant 1$. By Lemma 3 we have

$$a_n = \int_{-1}^{1} f(t) P_n(t) w(t) dt = \int_{-1}^{1} (f(t) - q_{n-1}(t)) P_n(t) w(t) dt,$$

where $q_{n-1}(t)$ is any polynomial of degree $n-1$. It follows that

$$|a_n| \leqslant E_{n-1}(f) \int_{-1}^{1} |P_n(t)| w(t) dt \leqslant E_{n-1}(f) \left(\int_{-1}^{1} w(t) dt \right)^{1/2},$$

where, in the notation of Lorentz [5],

$$E_{n-1}(f) = \inf_{q_{n-1}} \max_{-1 \leqslant t \leqslant 1} |f(t) - q_{n-1}(t)|.$$

Further, it is proved in [5, inequality (6), p. 78] that

$$E_{n-1}(f) \leqslant \frac{2R}{R-1} \cdot \frac{1}{R^n} \max_{u \in \gamma_R} |f(u)|.$$

Combining the above inequalities we obtain the desired result for $n \geqslant 1$. Finally, the case $n = 0$ of the Cauchy-type inequality is easily seen to be true since, for $P_0 = P_0(t)$, we have

$$|P_0| \left(\int_{-1}^{1} w(t) dt \right)^{1/2} = 1.$$

§3. *Proofs of Theorems 1, 2 and 3.* In the proofs of Theorems 1, 2 and 3, u and z are related by $u=\frac{1}{2}(z+z^{-1})$, $z=u+\sqrt{u^2-1}$ with $|z|>1$, the sign of the square root being chosen so that $|u+\sqrt{u^2-1}|>1$.

Proof of Theorems 1 and 2. We prove these two theorems together.

Sufficiency. We assume that

$$\begin{cases}\lim\limits_{n\to\infty} b_{nk}=b_k & \text{for} \quad k=0,1,\ldots;\\ M(p)=\sup\limits_{n\geqslant 0,k\geqslant 0}|b_{nk}|\left(\dfrac{p}{R}\right)^k<\infty & \text{for} \quad 1<p<P.\end{cases}$$

Let $\mathbf{a}\in\mathbf{A}_R$, or $\mathbf{a}\in\mathscr{E}_R$. For $1<p<P$ choose r so that $1<r<R$ and $p/r<P/R$. Now choose p_1 so that $p<p_1<P$ and $p/r=p_1/R$. Suppose $u\in D_p^\gamma$. Then $u=\frac{1}{2}(z+z^{-1})$ with $1\leqslant|z|<p$, and therefore, by Lemma 2,

$$\begin{aligned}|b_{nk}a_kP_k(u)|&\leqslant K(\varepsilon)|b_{nk}||a_k|(1+k)^{2+(2/\varepsilon)}p^k\\&=K(\varepsilon)|b_{nk}|\left(\frac{p}{r}\right)^k|a_k|(1+k)^{2+(2/\varepsilon)}r^k\\&=K(\varepsilon)|b_{nk}|\left(\frac{p_1}{R}\right)^k|a_k|(1+k)^{2+(2/\varepsilon)}r^k\\&\leqslant K(\varepsilon)M(p_1)|a_k|(1+k)^{2+(2/\varepsilon)}r^k<\infty.\end{aligned}$$

Further, by (i) (of either Theorem 1 or Theorem 2),

$$\lim_{n\to\infty} b_{nk}a_kP_k(u)=b_ka_kP_k(u).$$

Since $\sum_{k=0}^{\infty}|a_k|(1+k)^{2+(2/\varepsilon)}r^k<\infty$, and since p can be chosen arbitrarily close to P in $(1,P)$, it follows, by the Weierstrass M-test, that $g_n(u)$ exists for $n=0,1,\ldots$, and

$$\lim_{n\to\infty} g_n(u)=\lim_{n\to\infty}\sum_{k=0}^{\infty} b_{nk}a_kP_k(u)=\sum_{k=0}^{\infty} b_ka_kP_k(u)$$

on D_P^γ, and that the sequence (g_n) is uniformly convergent on compact subsets of D_P^γ. This completes the proof of the sufficiency of conditions (i) and (ii) both for Theorem 1 and Theorem 2.

Necessity. Let $a_k=1/R^k(k+1)^2$. Then $\mathbf{a}\in\mathbf{A}_R$ and $\mathbf{a}\in\mathscr{E}_R$. Under the hypotheses of either Theorem 1 or Theorem 2 the series

$$g_n(u)=\sum_{k=0}^{\infty} b_{nk}a_kP_k(u)$$

is convergent on D_P^γ and the sequence (g_n) is uniformly convergent on compact subsets of D_P^γ. Therefore, by the Weierstrass double-series theorem, (g_n)

converges to a holomorphic function on D_P^γ. By Lemma 3, we get, for the above sequence $\mathbf{a}$, that

$$b_{nk}a_k=\int_{-1}^{1} g_n(t)P_k(t)dt \qquad \text{for} \qquad n=0,1,\ldots.$$

Since $g_n(t)$ converges uniformly on $[-1,1]$ to $g(t)$ say, we get that

$$\lim_{n\to\infty} b_{nk}a_k=\int_{-1}^{1} g(t)P_k(t)dt=d_k.$$

Hence, for $k=0,1,\ldots,$

$$\lim_{n\to\infty} b_{nk}=b_k,$$

where $b_k=d_kR^k(k+1)^2$. This proves the necessity of condition (i) in both Theorem 1 and Theorem 2.

Suppose now that p and $\tilde p$ are fixed with $1<p<\tilde p<P$. Since $\mathbf{a}$ satisfies the hypotheses of both Theorem 1 and Theorem 2, the sequence (g_n) is uniformly convergent on $\bar D_{\tilde p}^\gamma$. Hence we have, for $u\in\bar D_{\tilde p}^\gamma$ and $n=0,1,\ldots,$ that $|g_n(u)|\leqslant M(\tilde p,\mathbf{a})<\infty$, $M(\tilde p,\mathbf{a})$ being independent of n. By Lemma 4 we get that

$$|b_{nk}a_k\tilde p^k|\leqslant c(\tilde p)M(\tilde p,\mathbf{a}) \qquad \text{for} \qquad n,k=0,1,\ldots.$$

Since $a_k=1/R^k(k+1)^2$, it follows that

$$|b_{nk}|\left(\frac{\tilde p}{R}\right)^k\frac{1}{(k+1)^2}\leqslant c(\tilde p)M(\tilde p,\mathbf{a}) \qquad \text{for} \qquad n,k=0,1,\ldots,$$

and hence that

$$\sup_{n\geqslant 0,k\geqslant 0}|b_{nk}|\left(\frac{p}{R}\right)^k\leqslant c(\tilde p)M(\tilde p,\mathbf{a})\sup_{k\geqslant 0}\left\{\left(\frac{p}{\tilde p}\right)^k(k+1)^2\right\}<\infty.$$

Therefore the condition

$$\sup_{n\geqslant 0,k\geqslant 0}|b_{nk}|\left(\frac{p}{R}\right)^k<\infty \qquad \text{whenever} \qquad 1<p<P,$$

is necessary, *i.e.*, condition (ii) is necessary in both Theorem 1 and Theorem 2.

Proof of Theorem 3. Sufficiency. We assume that

$$\begin{cases}\lim\limits_{n\to\infty} b_{nk}=b_k & \text{for} \quad k=0,1,\ldots;\\ M=\sup\limits_{n\geqslant 0,k\geqslant 0}|b_{nk}|^{1/(k+1)}<\infty.\end{cases}$$

Let $\mathbf{a}\in\mathscr{E}$, and let $u\in D_R^\gamma$. Then $u=\frac{1}{2}(z+z^{-1})$ with $1\leqslant|z|<R<\infty$, and so, by Lemma 2,

$$\begin{aligned}|b_{nk}a_kP_k(u)| &\leqslant K(\varepsilon)|b_{nk}||a_k|(1+k)^{2+(2/\varepsilon)}|z|^k\\ &\leqslant K(\varepsilon)|b_{nk}||a_k|(1+k)^{2+(2/\varepsilon)}R^k\\ &\leqslant K(\varepsilon)M|a_k|(1+k)^{2+(2/\varepsilon)}(MR)^k<\infty.\end{aligned}$$

From (i) we get

$$\lim_{n\to\infty} b_{nk}a_kP_k(u)=b_ka_kP_k(u).$$

Since $\sum_{k=0}^{\infty}|a_k(1+k)^{2+(2/\varepsilon)}(MR)^k<\infty$, and since R can be arbitrarily large, it follows, by the Weierstrass M-test, that $g_n(u)$ exists for $n=0, 1, \ldots$, and

$$\lim_{n\to\infty} g_n(u)=\lim_{n\to\infty}\sum_{k=0}^{\infty} b_{nk}a_kP_k(u)=\sum_{k=0}^{\infty} b_ka_kP_k(u)$$

on $\mathbb{C}$, and that the sequence (g_n) is uniformly convergent on compact subsets of $\mathbb{C}$.

Necessity. Let $a_k=k^{-k}$, so that $\mathbf{a}\in\mathscr{E}$. Then, by hypothesis, the series

$$g_n(u)=\sum_{k=0}^{\infty} b_{nk}a_kP_k(u)$$

is convergent on $\mathbb{C}$, and the sequence (g_n) is uniformly convergent on compact subsets of $\mathbb{C}$. By the Weierstrass double-series theorem, (g_n) converges to an entire function on $\mathbb{C}$. By Lemma 3 we have

$$b_{nk}a_k=\int_{-1}^{1} g_n(t)P_k(t)dt \qquad \text{for} \qquad n=0, 1, \ldots.$$

Since $g_n(t)$ is uniformly convergent on $[-1, 1]$ to $g(t)$ say, we get, for $k=0, 1, \ldots$, that

$$\lim_{n\to\infty} b_{nk}a_k=\int_{-1}^{1} g(t)P_k(t)dt=d_k,$$

and hence that

$$\lim_{n\to\infty} b_{nk}=b_k,$$

where $b_k=d_kk^k$ for $k=0, 1, 2, \ldots$. Thus condition (i) is necessary.

Suppose now that $\mathbf{a}$ is an arbitrary sequence in $\mathscr{E}$, and that $R>1$. Since the sequence (g_n) is uniformly convergent on $\bar{D}_R^\gamma$, we have, for $u\in\bar{D}_R^\gamma$ and $n=0, 1, \ldots$, that $|g_n(u)|\leqslant M(R, \mathbf{a})<\infty$. From Lemma 4 we get that

$$|b_{nk}a_k|\leqslant c(R)M(R, \mathbf{a})R^{-k} \qquad \text{for} \qquad n, k=0, 1, \ldots. \tag{1}$$

Hence $\sum_{k=0}^{\infty} b_{nk}a_k$ is convergent whenever $\mathbf{a}\in\mathscr{E}$, and we have, by Lemma 1, that

$$M_n=\sup_{k\geqslant 0}|b_{nk}|^{1/(k+1)}<\infty \qquad \text{for} \qquad n=0, 1, \ldots.$$

Assume now that

$$\sup_{n\geqslant 0}\sup_{k\geqslant 0}|b_{nk}|^{1/(k+1)}=\sup_{n\geqslant 0} M_n=\infty.$$

This implies that there exists a strictly increasing sequence of positive integers (n_j) such that $M_{n_j}\to\infty$. This in turn implies that there exists a sequence of non-negative integers (k_j) such that

$$|b_{n_j,k_j}|^{1/(k_j+1)}>\tfrac{1}{2}M_{n_j}\to\infty \qquad \text{as} \qquad j\to\infty. \tag{2}$$

We show now that the sequence (k_j) is not bounded. Assume that it is bounded. Then there is a positive integer k^* such that $0\leqslant k_j\leqslant k^*$. Since $\lim_{n\to\infty} b_{nk}=b_k$ for $k=0, 1, \ldots, k^*$, it follows that the set of numbers $(b_{nk})_{n\geqslant 0, 0\leqslant k\leqslant k^*}$ is bounded, and hence that the set of numbers $(|b_{nk}|^{1/(k+1)})_{n\geqslant 0, 0\leqslant k\leqslant k^*}$ is bounded. But this contradicts (2). Therefore the sequence (k_j) is not bounded. We can suppose (by considering a subsequence if necessary) that the sequence is strictly increasing. Choose

$$a_k=\begin{cases}\left(\dfrac{1}{|b_{n_j,k}|}\right)^{\frac{1}{2}(k+1)}, & \text{if } k=k_j,\\ 0, & \text{otherwise}.\end{cases}$$

We then have, by (2), that

$$|a_{k_j}|^{1/(k_j+1)}=\frac{1}{\sqrt{|b_{n_j,k_j}|}}<\left(\frac{1}{\frac{1}{2}M_{n_j}}\right)^{\frac{1}{2}(k_j+1)}\to 0 \qquad \text{as} \qquad j\to\infty.$$

Therefore $\mathbf{a}\in\mathscr{E}$, but

$$|b_{n_j,k_j}|a_{k_j}=\sqrt{|b_{n_j,k_j}|}\to\infty \qquad \text{as} \qquad j\to\infty,$$

which contradicts (1). Thus the condition

$$\sup_{n\geqslant 0, k\geqslant 0}|b_{nk}|^{1/(k+1)}<\infty$$

is necessary, *i.e.*, condition (ii) is necessary.

§4. *Additional Theorems.* In this section we prove some theorems showing that the ellipse of convergence D_P^γ specified in Theorem 2 cannot be enlarged when the matrix $\mathbf{B}$ satisfies conditions (i) and (ii) of that theorem together with certain other conditions. Analogous theorems concerning matrix transformations of power series appear in [1].

THEOREM 4. *Suppose that P and R are finite numbers greater than 1, and that $\mathbf{B}\equiv(b_{nk})$ is a triangular infinite matrix (i.e., $b_{nk}=0$ for $k>n$) satisfying*

$$M(p)=\sup_{n\geqslant 0, k\geqslant 0}|b_{nk}|\left(\frac{p}{R}\right)^k<\infty \qquad \textit{for} \qquad 1<p<P.$$

Then, for each $\mathbf{a}\in\mathbf{A}_R$ *and each* $R_1\geqslant P$,

$$\limsup_{n\to\infty}\max_{u\in\gamma_{R_1}}\left|\sum_{k=0}^{n} b_{nk}a_kP_k(u)\right|^{1/n}\leqslant\frac{R_1}{P}.$$

Proof. Choose $R_1\geqslant P>1$, and suppose $\mathbf{a}\in\mathbf{A}_R$. Let $1/P<\lambda<1$, and take $p=\lambda P>1$. Then $1<p<P$. Since $\limsup|a_k|^{1/(k+1)}=1/R$, there is a positive constant $c(\lambda)$ such that

$$|a_k|\leqslant\frac{c(\lambda)}{(\lambda R)^k}\qquad\text{for}\qquad k\geqslant 0.$$

By Lemma 2, for $u\in\gamma_{R_1}$ we have $|P_k(u)|\leqslant K(\varepsilon)(1+k)^{2+(2/\varepsilon)}R_1^k$ and hence

$$\begin{aligned}\left|\sum_{k=0}^{n} b_{nk}a_kP_k(u)\right|&\leqslant K(\varepsilon)\sum_{k=0}^{n}|b_{nk}|\left(\frac{p}{R}\right)^k|a_k|R^k\left(\frac{R_1}{p}\right)^k(1+k)^{2+(2/\varepsilon)}\\&\leqslant K(\varepsilon)M(p)c(\lambda)\sum_{k=0}^{n}\left(\frac{R}{\lambda R}\right)^k\left(\frac{R_1}{\lambda P}\right)^k(1+k)^{2+(2/\varepsilon)}\\&\leqslant K(\varepsilon)M(p)c(\lambda)(1+n)^{2+(2/\varepsilon)}\sum_{k=0}^{n}\left(\frac{R_1}{\lambda^2P}\right)^k.\end{aligned}$$

Since $R_1/\lambda^2P>R_1/P\geqslant 1$, it follows that

$$\limsup_{n\to\infty}\max_{u\in\gamma_{R_1}}\left|\sum_{k=0}^{n} b_{nk}a_kP_k(u)\right|^{1/n}\leqslant\lim_{n\to\infty}\left(\sum_{k=0}^{n}\left(\frac{R_1}{\lambda^2P}\right)^k\right)^{1/n}=\frac{R_1}{\lambda^2P}.$$

Letting $\lambda\nearrow 1$ we get

$$\limsup_{n\to\infty}\max_{u\in\gamma_{R_1}}\left|\sum_{k=0}^{n} b_{nk}a_kP_k(u)\right|^{1/n}\leqslant\frac{R_1}{P}.$$

Remark. Assume that a triangular matrix $\mathbf{B}$ satisfies

$$M(p)=\sup_{n\geqslant 0,k\geqslant 0}|b_{nk}|\left(\frac{p}{R}\right)^k<\infty\qquad\text{for}\qquad 1<p<P.$$

Then

$$|b_{nn}|^{1/n}\frac{p}{R}\leqslant M(p)^{1/n}\to 1\qquad\text{as}\qquad n\to\infty,$$

and hence

$$\limsup_{n\to\infty}|b_{nn}|^{1/n}\leqslant\frac{R}{p}\qquad\text{for each}\quad p\in(1,P).$$

Letting $p\nearrow P$ we get

$$\limsup_{n\to\infty}|b_{nn}|^{1/n}\leqslant\frac{R}{P}.$$

This suggests that it is not inappropriate to impose the condition

$$\lim_{n\to\infty}|b_{nn}|^{1/n}=\frac{R}{P},$$

as we do in the following theorem.

THEOREM 5. *Let* $\mathbf{B}$ *be a triangular matrix. Suppose that*

$$\lim_{n\to\infty}|b_{nn}|^{1/n}=\frac{R}{P},$$

where P *and* R *are finite numbers greater than* 1. *Then for each* $\mathbf{a}\in\mathbf{A}_R$ *and each* $R_1\geqslant P$ *we have*

$$\limsup_{n\to\infty}\max_{u\in\gamma_{R_1}}\left|\sum_{k=0}^{n} b_{nk}a_kP_k(u)\right|^{1/n}\geqslant\frac{R_1}{P}.$$

Proof. Assume that the conclusion of the theorem is not true. Then there is an $\mathbf{a}^*\in\mathbf{A}_R$ and an $R_1\geqslant P>1$ such that

$$\limsup_{n\to\infty}\max_{u\in\gamma_{R_1}}\left|\sum_{k=0}^{n} b_{nk}a_k^*P_k(u)\right|^{1/n}<\frac{R_1}{P}.$$

Therefore there exists a number $\tilde{R}$ such that $1<\tilde{R}<R_1$ and, for all n sufficiently large,

$$\max_{u\in\gamma_{R_1}}\left|\sum_{k=0}^{n} b_{nk}a_k^*P_k(u)\right|^{1/n}\leqslant\frac{\tilde{R}}{P},\qquad\text{and hence}\qquad\max_{u\in\gamma_{R_1}}\left|\sum_{k=0}^{n} b_{nk}a_k^*P_k(u)\right|\leqslant\left(\frac{\tilde{R}}{P}\right)^n.$$

Applying Lemma 4 to the function $g_n(u)=\sum_{k=0}^{n}b_{nk}a_k^*P_k(u)$ we get in particular that, for all large n,

$$|b_{nn}||a_n^*|R_1^n\leqslant c(R_1)\left(\frac{\tilde{R}}{P}\right)^n,$$

and therefore

$$|b_{nn}|^{1/n}|a_n^*|^{1/n}R_1\leqslant c(R_1)^{1/n}\frac{\tilde{R}}{P}.$$

From the last inequality we get that

$$\frac{\tilde{R}}{P}\geqslant\limsup_{n\to\infty}(|b_{nn}|^{1/n}|a_n^*|^{1/n}R_1)=R_1\lim_{n\to\infty}|b_{nn}|^{1/n}\,.\,\limsup_{n\to\infty}|a_n^*|^{1/n}=\frac{R_1}{P}.$$

But this is a contradiction since $1<\tilde{R}<R_1$. Hence the conclusion of the theorem must hold.

The next two theorems are analogues of Theorems 6 and 7 (concerning matrix transformations of power series) in [**1**], which in turn generalize results about regular and non-regular Nörlund matrices due respectively to Luh [**6**] and K. Stadtmüller [9, Theorems 6 and 7]. The first of these new theorems, which follows immediately from Theorems 4 and 5, shows, *inter alia*, that the sequence (g_n) specified in Theorem 2 cannot converge uniformly in the interior of any ellipse γ_{P_1} with $P_1 > P$ when **B** is a triangular matrix satisfying condition (ii) of Theorem 2 together with the diagonal condition of Theorem 5.

THEOREM 6. *Suppose that P and R are finite numbers greater than* 1, *and that* **B** *is a triangular matrix satisfying*

$$M(p) = \sup_{n\geqslant 0, k\geqslant 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty \quad \textit{for} \quad 1 < p < P,$$

and

$$\lim_{n\to\infty} |b_{nn}|^{1/n} = \frac{R}{P}.$$

Then, for each $\mathbf{a} \in \mathbf{A}_R$ *and each* $R_1 \geqslant P$,

$$\limsup_{n\to\infty} \max_{u\in\gamma_{R_1}} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^{1/n} = \frac{R_1}{P}.$$

The next theorem shows that the ellipse γ_{R_1} in the conclusion of Theorem 6 can be replaced by any arc of that ellipse (provided condition (i) of Theorem 2 is also satisfied when $R_1 = P$).

THEOREM 7. *Suppose that P and R are finite numbers greater than* 1, *and that* **B** *is a triangular matrix such that*

$$M(p) = \sup_{n\geqslant 0, k\geqslant 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty \quad \textit{for} \quad 1 < p < P,$$

and

$$\lim_{n\to\infty} |b_{nn}|^{1/n} = \frac{R}{P}.$$

(i) *Then, for each* $\mathbf{a} \in \mathbf{A}_R$ *and each* $R_1 > P$,

$$\limsup_{n\to\infty} \max_{u\in\Gamma} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^{1/n} = \frac{R_1}{P},$$

where Γ *is any closed non-trivial arc of* γ_{R_1}.

(ii) *If, in addition,*

$$\lim_{n\to\infty} b_{nk} = b_k \quad \textit{for} \quad k = 0, 1, \ldots, \quad \textit{where} \quad b_k \neq 0 \quad \textit{for} \quad k > k^*,$$

then, for each $\mathbf{a} \in \mathbf{A}_R$,

$$\limsup_{n\to\infty} \max_{u\in\Gamma} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^{1/n} = 1,$$

where Γ *is any closed non-trivial arc of* γ_P.

Proof of (i). By Theorem 6 we know that

$$\limsup_{n\to\infty} \max_{u\in\Gamma} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^{1/n} \leqslant \frac{R_1}{P}.$$

Hence it is enough to prove that, for every $\mathbf{a} \in \mathbf{A}_R$,

$$\limsup_{n\to\infty} \max_{u\in\Gamma} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^{1/n} \geqslant \frac{R_1}{P}, \tag{3}$$

which we now proceed to do. Assume that (3) is not true. Then there exists a sequence $\mathbf{a}^* \in \mathbf{A}_R$ and a number $\tilde{R}$ such that $P < \tilde{R} < R_1$ and

$$\limsup_{n\to\infty} \max_{u\in\Gamma} |g_n(u, \mathbf{a}^*)|^{1/n} \leqslant \frac{\tilde{R}}{P}.$$

Hence given $\varepsilon > 0$ we have, for $z = u + \sqrt{u^2 - 1}$ and all sufficiently large n,

$$\max_{u\in\Gamma} \left| \frac{g_n(u, \mathbf{a}^*)}{z^n} \right| \leqslant \left(\frac{\tilde{R}}{P} \cdot \frac{1}{R_1}\right)^n 2^{\varepsilon n} = \left(\frac{\tilde{R}}{R_1}\right)^n \left(\frac{2^\varepsilon}{P}\right)^n.$$

Further, from Theorem 6 we get that, for all large n,

$$\max_{u\in\gamma_P} \left| \frac{g_n(u, \mathbf{a}^*)}{z^n} \right| \leqslant \left(\frac{2^\varepsilon}{P}\right)^n$$

and

$$\max_{u\in\gamma_{R_1}} \left| \frac{g_n(u, \mathbf{a}^*)}{z^n} \right| \leqslant \left(\frac{2^\varepsilon}{P}\right)^n.$$

Let $P < r < R_1$. Since the function $z = u + \sqrt{u^2 - 1}$ is holomorphic and different from zero on $\mathbb{C}\setminus[-1, 1]$, we have, by Nevanlinna's N-constants theorem (see [3, Theorem 18.3.3]), that there exist positive constants $\theta_1, \theta_2, \theta_3$ (depending on r but not on ε) such that $\theta_1 + \theta_2 + \theta_3 = 1$ and

$$\max_{u\in\gamma_r} \left| \frac{g_n(u, \mathbf{a}^*)}{z^n} \right| \leqslant \left(\frac{\tilde{R}}{R_1}\frac{2^\varepsilon}{P}\right)^{n\theta_1} \left(\frac{2^\varepsilon}{P}\right)^{n\theta_2} \left(\frac{2^\varepsilon}{P}\right)^{n\theta_3} = \left(\frac{\tilde{R}}{R_1}\right)^{n\theta_1} \left(\frac{2^\varepsilon}{P}\right)^n$$

for all sufficiently large n. Hence, choosing $\varepsilon > 0$ so small that $(\tilde{R}/R_1)^{\theta_1} 2^\varepsilon < 1$, we get

$$\limsup_{n\to\infty} \max_{u\in\gamma_r} |g_n(u, \mathbf{a}^*)|^{1/n} \leqslant \left(\frac{\tilde{R}}{R_1}\right)^{\theta_1} 2^\varepsilon \frac{r}{P} < \frac{r}{P}.$$

Since $r > P$, the last inequality contradicts the conclusion of Theorem 5. Hence (3) must hold when $R_1 > P$.

Proof of (ii). By Theorem 6 we know in this case that

$$\limsup_{n\to\infty} \max_{u\in\Gamma} \left|\sum_{k=0}^{n} b_{nk}a_k P_k(u)\right|^{1/n} \leqslant 1.$$

Hence it is enough to prove that, for every $\mathbf{a}\in \mathbf{A}_R$,

$$\limsup_{n\to\infty} \max_{u\in\Gamma} \left|\sum_{k=0}^{n} b_{nk}a_k P_k(u)\right|^{1/n} \geqslant 1, \tag{4}$$

Suppose (4) is not true. Then for some $\mathbf{a}^*\in \mathbf{A}_R$ we have

$$\limsup_{n\to\infty} \max_{u\in\Gamma} \left|\sum_{k=0}^{n} b_{nk}a_k^* P_k(u)\right|^{1/n} < 1.$$

Write

$$g_n(u, \mathbf{a}^*) = \sum_{k=0}^{n} b_{nk}a_k^* P_k(u).$$

It follows that there exists a positive number $q < R_1/P = 1$, such that, for all n sufficiently large,

$$\sup_{u\in\Gamma} |g_n(u, \mathbf{a}^*)| < q^n.$$

Given $\alpha > 0$ we get from Theorem 6 that, for all n sufficiently large,

$$\max_{u\in\gamma_P} |g_n(u, \mathbf{a}^*)| \leqslant 2^{\alpha n}.$$

By Nevanlinna's N-constants theorem, there exists a positive number $\theta < 1$ (independent of α) such that, for all large n,

$$\max_{-1\leqslant u\leqslant 1} |g_n(u, \mathbf{a}^*)| \leqslant (q^\theta 2^{(1-\theta)\alpha})^n.$$

Since we can choose $\alpha > 0$ so small that $q^\theta 2^{(1-\theta)\alpha} < 1$, it follows that

$$\max_{-1\leqslant u\leqslant 1} |g_n(u, \mathbf{a}^*)| \to 0 \quad \text{as} \quad n\to\infty.$$

By Lemma 3 we have

$$b_{nk}a_k = \int_{-1}^{1} g_n(t, \mathbf{a}^*) P_k(t)\,dt \qquad \text{for} \qquad n = 0, 1, \ldots.$$

Since $g_n(t, \mathbf{a}^*)$ tends uniformly to 0 on $[-1, 1]$ as $n\to\infty$, it follows that

$$0 = \lim_{n\to\infty} b_{nk}a_k^* = b_k a_k^* \qquad \text{for} \qquad k = 0, 1, \ldots.$$

Since $\mathbf{a}^*\in\mathbf{A}_R$ we have that $a_k^* \neq 0$ for some $k > k^*$. Hence $b_k = 0$ for such a k. But this contradicts the assumption that $b_k \neq 0$ for $k > k^*$. Therefore (4) must hold.

§5. *Chebyshev Polynomials.* In this section we restrict (P_n) to be the orthonormal sequence on [0, 1] of Chebyshev polynomials of the first or second kind, the corresponding weight functions of which are respectively $w(x) = \frac{1}{2}\pi(1-x^2)^{-1/2}$ and $w(x) = \frac{1}{2}\pi(1-x^2)^{1/2}$. The special properties of these Chebyshev polynomials that makes them amenable to the proof of Theorem 8 (below) are the familiar identities

$$2P_n(\tfrac{1}{2}(z+z^{-1})) = z^n + z^{-n} \tag{5}$$

when P_n is of the first kind, and

$$(z - z^{-1})P_n(\tfrac{1}{2}(z+z^{-1})) = z^{n+1} - z^{-n-1} \tag{6}$$

when P_n is of the second kind.

The said theorem deals with the possibility of pointwise convergence of the sequence $(g_n(u))$ specified in Theorem 2 outside the convergence ellipse γ_P. Its analogue for power series is Theorem 8 in [**1**], which generalizes results due to Lejá [**4**] and Stadtmüller [**9**, Theorem 8] about regular and non-regular Nörlund matrices respectively.

THEOREM 8. *Suppose that P and R are finite numbers greater than* 1, *and that* **B** *is a triangular matrix such that*

(i) $\lim_{n\to\infty} b_{nk} = b_k$ *for* $k = 0, 1, \ldots$ *where* $b_k \neq 0$ *for* $k > k^*$;

(ii) $M(p) = \sup_{n\geqslant 0, k\geqslant 0} |b_{nk}|(p/R)^k < \infty$ *for* $1 < p < P$; $\lim_{n\to\infty} |b_{nn}|^{1/n} = R/P$; *and*

(iii) $|b_{nk}| \leqslant c(\tilde{R})|b_{nn}|(P/\tilde{R})^{n-k}$ *for* $1 < \tilde{R} < R$ *and* $0 \leqslant k \leqslant n$.

Suppose that $\mathbf{a}\in\mathbf{A}_R$ *and that* $\limsup_{n\to\infty} |a_n|R^n > 0$. *Let*

$$g_n(u) = \sum_{k=0}^{n} b_{nk}a_k P_k(u),$$

where (P_k) *is the orthonormal sequence on* $[-1, 1]$ *of Chebyshev polynomials of the first or second kind, and let* $P_1 > P$. *Then* $\limsup_{n\to\infty} |g_n(u)|^{1/n} \leqslant 1$ *for at most a finite number of points u outside the ellipse* γ_{P_1} *and hence, in particular, the sequence* (g_n) *can converge at most at a finite number of points u outside the ellipse* γ_{P_1}.

Proof. Assume that u is a point outside the ellipse γ_{P_1} for which

$$\limsup_{n\to\infty} |g_n(u)|^{1/n} \leqslant 1. \tag{7}$$

Let $z = u + \sqrt{u^2 - 1}$, so that $|z| > P_1$; and let

$$\tilde{g}_n(z) = \sum_{k=0}^{n} b_{nk}a_k z^k.$$

Then, by (5),

$$2g_n(u) = 2\sum_{k=0}^{n} b_{nk}a_k P_k(u) = \tilde{g}_n(z) + \tilde{g}_n(z^{-1})$$

when the Chebyshev polynomials P_k are of the first kind; and, by (6),

$$(z-z^{-1})g_n(u)=z\tilde{g}_n(z)-z^{-1}\tilde{g}_n(z^{-1})$$

when the Chebyshev polynomials P_k are of the second kind.

Since $|z^{-1}|<P_1^{-1}<P$ it follows from Theorem 2 in [1] that $\tilde{g}_n(z^{-1})$ tends to a finite limit as $n\to\infty$, and therefore from (7) that, in either case,

$$\limsup_{n\to\infty}|\tilde{g}_n(z)|^{1/n}\leqslant 1. \tag{8}$$

Theorem 8 in [1] tells us that inequality (8) can hold for at most a finite number of points z satisfying $|z|>P_1$, and thus (7) can hold for at most finitely many points u outside the ellipse γ_{P_1}.

Remarks. A Nörlund matrix $\mathbf{N}_B$ for which

$$\lim_{n\to\infty}\frac{B_{n-1}}{B_n}=b \quad \text{with} \quad |b|=\frac{R}{P}$$

satisfies all the conditions on the matrix in Theorem 8. In this case, however, the condition $\limsup|a_n|R^n>0$ can be omitted since the corresponding version of the theorem for power series has recently been proved by K. Stadtmüller and Grosse-Erdmann [10, Remark 3.7].

An open and challenging question is whether Theorem 8 holds for other orthogonal polynomials.

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