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Weighted Means and Summability by Generalized Nörlund and Other Methods*

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It is proved that if the weighted means of a sequence satisfy certain order conditions, then the sequence is summable by every method of a family of methods Γ_p based on a given sequence (p_n). The family Γ_p includes the power series method J_p and the generalized Nörlund method (N, p, p). © 1994 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

Suppose throughout that (s_n) is a given sequence, and that (q_n) is a sequence of positive numbers. The sequence of weighted means (t_n) is defined by

$$t_n := \frac{1}{Q_n} \sum_{k=0}^{n} q_k s_k, \qquad \text{where} \quad Q_n := \sum_{k=0}^{n} q_k.$$

The sequence (s_n) is said to be summable to s by the weighted mean method M_q, and we write

$$s_n \to s(M_q) \qquad \text{if} \quad t_n \to s.$$

In particular, the Cesàro method C_1 and the logarithmic method l are the

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methods M_q with $q_k=1$ and $q_k=1/(k+1)$, respectively. The sequence of C_1-means of any sequence (x_n) will be denoted by (x_n^1), so that

$$x_n^1=\frac{1}{n+1}\sum_{k=0}^{n} x_k.$$

Suppose further that (p_n) is a sequence of real non-negative numbers with $p_0>0$, and that the power series

$$p(t):=\sum_{n=0}^{\infty} p_n t^n$$

has radius of convergence $R>0$. Then the power series method of summability J_p is defined as follows.

$s_n \to s(J_p)$ if

$$p_s(t):=\sum_{n=0}^{\infty} s_n p_n t^n$$

is convergent for $|t|<R$, and if

$$\sigma_p(t):=\frac{p_s(t)}{p(t)} \to s \qquad \text{as} \quad t\to R-.$$

In this note we impose certain smoothness conditions on the sequence (p_n), namely

$$p_n:=e^{-g(n)} \quad \text{for } n\geqslant x_0\in\mathbb{N}, \qquad \text{and} \qquad p_n:=p_{x_0} \quad \text{for } 0\leqslant n\leqslant x_0, \tag{1.1}$$

where the function g has the properties

$$\begin{cases} g''(x) \text{ is continuous, positive, and decreasing to 0 on } [x_0,\infty), \\ \text{while } x^2g''(x) \text{ is increasing to } \infty \text{ on } [x_0,\infty). \end{cases} \tag{C}$$

Our treatment will cover both Abel-type $(R<\infty)$ and Borel-type $(R=\infty)$ methods. In particular it will include the Borel-type method (B,α,β), $\alpha>0$, for which $g(x)=\log\Gamma(\alpha x+\beta)$.

By [5, Theorem 5], the J_p method is regular when $R=\infty$, that is, $s_n\to s$ implies $s_n\to s(J_p)$. Suppose, therefore, that $R<\infty$. Then, by (1.1) and (C), we have that $g'(x)$ increases and tends to ρ (say) as $x\to\infty$, and consequently that $g(n)/n\to\rho$ as $n\to\infty$. It follows, by the nth root test, that $R=e^\rho$, and hence that $g(n)-g(x_0)\leqslant n\log R$ for $n\geqslant x_0$, whence

$$\sum_{n=0}^{\infty} p_nR^n \geqslant \sum_{n=x_0}^{\infty} R^{-n}R^ne^{-g(x_0)}=\infty.$$

Thus, by [5, Theorem 5], J_p is also regular when $R<\infty$.

We define a function ϕ by

$$\phi(x):=\frac{1}{\sqrt{g''(x)}} \quad \text{for } x\geqslant x_0, \qquad \text{and} \qquad \phi(x):=\phi(x_0) \quad \text{for } 0\leqslant x\leqslant x_0. \tag{1.2}$$

We also consider the scale of generalized Nörlund summability methods $(N,p^{*\alpha},p)$, $\alpha=1,2,...$, defined as (see [1, 6])

$$s_n\to s(N,p^{*\alpha},p) \qquad \text{if} \quad \sigma_n^\alpha:=\frac{1}{p_n^{*(\alpha+1)}}\sum_{k=0}^{n} p_{n-k}^{*\alpha}p_ks_k\to s \qquad \text{as} \quad n\to\infty,$$

where

$$p_n^{*1}:=p_n, \qquad \text{and} \qquad p_n^{*(\alpha+1)}:=\sum_{k=0}^{\infty} p_{n-k}^{*\alpha}p_k>0.$$

Let Γ_p denote the family of all the methods $(N,p^{*\alpha},p)$ together with J_p. It is known [6, Theorem 1] that

$$(N,p^{*\alpha},p)\subseteq(N,p^{*\beta},p)\subseteq J_p \qquad \text{for} \quad \alpha<\beta, \quad \alpha,\beta\in\mathbb{N}, \tag{1.3}$$

that is, the methods in (1.3) increase in strength from left to right. It follows from (1.3) and the regularity of J_p that all the methods in Γ_p are regular when (1.1) and (C) hold. The most familiar family Γ_p is generated by $p_n=1/n!=e^{-\log\Gamma(n+1)}$. In this family J_p is the Borel method B, and $(N,p^{*\alpha},p)$ is the Euler–Knopp method $E_{(\alpha-1)/\alpha}$ for which $\sigma_n^\alpha=(1/\alpha^n)\sum_{k=0}^{n}\binom{n}{k}(\alpha-1)^k s_k$.

In general C_1-summability does not imply summability by any member of Γ_p. However, the following result, which we prove in Section 3, shows that C_1-summability together with an order condition does imply summability by every member of Γ_p.

THEOREM C. *Suppose that the sequence* (p_n) *is given by* (1.1) *with* g *satisfying* (C), *and that*

$$s_n^1=s+o\left(\frac{\phi(n)}{n}\right),$$

where ϕ *is given by* (1.2). *Then* (s_n) *is summable to* s *by every member of* Γ_p.

This theorem has been proved ([6, Theorem 2]; see also [10]) for a sequence (p_n) such that $p_n\sim e^{-g(n)}$ and p_n/p_{n+1} is non-decreasing, with g

satisfying (C). The special case $p_n = 1/n!$, for which $\phi(n)/n \sim 1/\sqrt{n}$, is a classical result of Knopp's (see [5, Theorem 149]).

We also establish the following two theorems.

THEOREM 1. *Suppose that the sequence (p_n) is given by (1.1) with g satisfying (C), and in addition*

$$x^{\varepsilon} g''(x) \text{ is decreasing on } [x_0, \infty) \text{ for some } \varepsilon > 0. \tag{1.4}$$

Suppose also that

$$\frac{q_{n-1}}{q_n} = 1 + O(n^{-1}) \tag{1.5}$$

and

$$t_n = s + \frac{\mu}{Q_n} + o\left(\frac{\phi(n)\, q_n}{Q_n}\right), \tag{1.6}$$

where s and μ are constants.

Then (i) *(s_n) is summable to s by every member of Γ_p (and by C_1).*

Moreover, (ii) *"o" cannot be replaced by "O" in* (1.6).

Indeed, (iii) *there exists a C_1-summable bounded sequence (s_n) which is not summable by any member of Γ_p and which satisfies*

$$t_n = O\left(\frac{\phi(n)\, q_n}{Q_n}\right).$$

Theorem 1 generalizes [2, Theorem 1 which deals with $p_n = 1/n!$], which in turn generalized [11, Theorem 1 which deals with $p_n = 1/n!$, $q_n = 1/(n+1)$]. Theorem 1 shows that M_q-summability together with an order condition implies summability by every member of Γ_p and that the order condition is best possible in a strong sense.

THEOREM 2. *Suppose that the sequence (p_n) is given by (1.1) with g satisfying (C), (1.5), and in addition*

$$\liminf n g''(n) > 0. \tag{1.7}$$

Suppose also that condition (1.4) holds, and that

$$n q_n s_n = O_L(1), \tag{1.8}$$

$$\frac{1}{n q_n} = O(1), \tag{1.9}$$

and

$$\frac{1}{n} \sum_{k=0}^{n} \frac{k}{\phi(k)}\, t_k Q_k = \mu + o\left(\frac{\phi(n)}{n}\right), \tag{1.10}$$

where μ is a constant. Then (s_n) is summable to 0 by every member of Γ_p.

The notation $x_n = O_L(1)$ signifies, as usual, that $\liminf x_n > -\infty$. Theorem 2 generalizes [2, Theorem 2 which deals with $p_n = 1/n!$], which in turn generalized [11, Theorem 2 which deals with $p_n = 1/n!$, $q_n = 1(n+1)$].

2. PRELIMINARY RESULTS

In all that follows we assume that

$$p_n := e^{-g(n)} \quad \text{for} \quad x_0 \leqslant n \in \mathbb{N},$$

and define

$$p_x := e^{-g(x)} \text{ for } x \geqslant x_0, \qquad \text{and} \qquad p_x := p_{x_0} \text{ for } 0 \leqslant x \leqslant x_0,$$

even when x is not an integer.

LEMMA 1. *Suppose that the sequence (p_n) is given by (1.1) with g satisfying (C). Then*

(i) $p_{n-x} p_x \leqslant p_{n-x-1} p_{x+1}$ *for* $0 \leqslant x \leqslant \frac{1}{2}n - 1$, *and* $\max_{0 \leqslant x \leqslant n} p_{n-x} p_x = p_{n/2}^2$, *and*

(ii) $p_n^{*2} \sim \sqrt{\pi}\, \phi(n/2)\, p_{n/2}^2$.

Proof. (i) Since $g''(x) > 0$ for $x \geqslant x_0$, we have that

$$\frac{d}{dx}(p_{n-x} p_x) = p_{n-x} p_x (g'(n-x) - g'(x))$$

is positive for $x_0 \leqslant x < \frac{1}{2}n$ and negative for $x_0 \leqslant \frac{1}{2}n < x \leqslant n - x_0$. The desired conclusions follow.

(ii) It follows from the monotonicity of $g''(x)$ and $x^2 g''(x)$ that if $x_0 \leqslant \frac{3}{4}x \leqslant t \leqslant x$, then

$$0 \leqslant \frac{g''(t)}{g''(x)} - 1 = \frac{t^2 g''(t)}{x^2 g''(x)} \cdot \frac{x^2}{t^2} - 1 \leqslant \frac{x^2}{t^2} - 1 \leqslant \frac{28}{9} \cdot \frac{x-t}{x},$$

while if $x_0 \leqslant x \leqslant t \leqslant \frac{5}{4}x$, then

$$0 \leqslant 1 - \frac{g''(t)}{g''(x)} = 1 - \frac{t^2 g''(t)}{x^2 g''(x)} \cdot \frac{x^2}{t^2} \leqslant 1 - \frac{x^2}{t^2} \leqslant 1 - \frac{x^2}{t^2} \leqslant \frac{45}{16} \cdot \frac{t-x}{x}.$$

Hence

$$\left|\frac{g''(t)}{g''(x)}-1\right|\leqslant 4\,\frac{|t-x|}{x} \qquad \text{whenever} \quad x,\, t\geqslant x_0 \text{ and } |t-x|\leqslant\frac{x}{4}. \tag{2.1}$$

Further, for $x_0\leqslant x\leqslant\frac{1}{2}n$,

$$\frac{p_{n-x}\,p_x}{p_{n/2}^2}=e^{-h(n,\,x)}, \qquad \text{where} \quad h(n,x):=g(n-x)-2g(n/2)+g(x). \tag{2.2}$$

By the second mean value theorem, we have

$$h(n,x)=(x-n/2)^2\,g''(\xi) \qquad \text{with} \quad x_0\leqslant x\leqslant\xi\leqslant n-x. \tag{2.3}$$

In view of (2.1), (2.2), and (2.3), and since $\phi(n/2)\to\infty$, we can now apply standard techniques (as in [3, 6, 7, 8]) to obtain

$$\begin{aligned}\frac{p_n^{*2}}{p_{n/2}^2}&=\int_{3n/8}^{5n/8} e^{-h(n,x)}\,dx+o(\phi(n/2))\\&=\int_{3n/8}^{5n/8} e^{-(x-n/2)^2 g''(n/2)}\,dx+o(\phi(n/2))\\&=\int_{-\infty}^{\infty} e^{-(x-n/2)^2 g''(n/2)}\,dx+o(\phi(n/2))\\&=\sqrt{\pi}\,\phi(n/2)(1+o(1)) \quad \text{as} \quad n\to\infty. \;\blacksquare\end{aligned}$$

LEMMA 2. *Suppose that the sequence* (p_n) *is given by* (1.1) *with* g *satisfying* (C). *Let*

$$\begin{cases}\Delta_n:=p(u_n)\,u_n^{-n}, & \text{with}\\ u_n:=e^{g'(n)} \text{ for } n\geqslant x_0, & \text{and} \quad u_n:=u_{x_0} \text{ for } 0\leqslant n\leqslant x_0.\end{cases} \tag{2.4}$$

Then

(i) $u_{n+1}\geqslant u_n$, *and* $u_n\to R$;

(ii) $p(u_n)=\sqrt{2\pi}\,\phi(n)\,e^{-g(n)+ng'(n)}(1+o(1))$; *and*

(iii) *for every* $\delta>0$ *there is an* $n_0\in\mathbb{N}$ *such that*

$$\sum_{n<\nu\leqslant n+\delta\phi(n)}\frac{p_\nu}{\Delta_\nu}\geqslant\frac{\delta}{4\sqrt{2\pi}} \qquad \textit{for} \quad n\geqslant n_0.$$

Proof. (i) This follows from (2.4) since $g'(n)$ increases for $n\geqslant x_0$ and (as in the proof of the regularity of J_p in Section 1) $e^{g'(n)}\to R$.

(ii) This is an immediate consequence of [4, Lemma 2].

(iii) It follows from (ii) that, for n sufficiently large,

$$\begin{aligned}\sum_{n<\nu\leqslant n+\delta\phi(n)}\frac{p_\nu}{\Delta_\nu}&\geqslant\frac{1}{2\sqrt{2\pi}}\sum_{n<\nu\leqslant n+\delta\phi(n)}\frac{\nu}{\phi(\nu)}\cdot\frac{1}{\nu}\\&\geqslant\frac{\delta\phi(n)}{2\sqrt{2\pi}}\cdot\frac{n}{\phi(n)}\cdot\frac{1}{n+\delta\phi(n)}\geqslant\frac{\delta}{4\sqrt{2\pi}},\end{aligned}$$

since $\phi(n)/n$ decreases to 0. ▮

For the purposes of the next lemma we recall the following definition given by Lorentz [9].

DEFINITION. The characteristic function $\omega(n)$ of a (finite or infinite) sequence $n_1<n_2<\cdots$ of positive integers is defined for all $n\geqslant 0$ as the number of n_ν satisfying the inequality $n_\nu\leqslant n$.

Let $\Omega(n)$ be a positive non-decreasing function defined for $n\geqslant 0$ and tending to ∞ as $n\to\infty$. For any such function the class $\Theta_1(\Omega)$ consists of all real bounded sequences (s_n) for which the set of indices $n_1<n_2<\cdots$ with non-vanishing s_n has characteristic function $\omega(n)\leqslant\Omega(n)$. The class $\Theta_2(n)$ consists of all real sequences (s_n) such that the sums $s_0+s_1+\cdots+s_n=O(\Omega(n))$.

The function $\Omega(n)$ is a summability function of the first or second kind for a summability method P, if all sequences in $\Theta_1(\Omega)$ or $\Theta_2(\Omega)$, respectively, are P-summable.

LEMMA 3. *Suppose that the sequence* (p_n) *is given by* (1.1) *with* g *satisfying* (C), *and that* $\Omega(n)\neq o(\phi(n))$. *Then* $\Omega(n)$ *is not a summability function of either kind for* J_p.

Proof. We consider the regular summability method P defined as

$$s_n\to s(P) \qquad \text{if} \quad \frac{1}{p(u_n)}\sum_{\nu=0}^{\infty}u_n^\nu p_\nu s_\nu\to s \quad \text{as } n\to\infty,$$

where u_n is given by (2.4). Then $J_p\subseteq P$.

Let $A(m;\Omega)$ denote the least upper bound of

$$\frac{1}{p(u_m)}\sum_{\nu=0}^{\infty}u_m^{n_\nu}p_{n_\nu}$$

for all sequences (n_ν) with $\omega(n)\leqslant\Omega(n)$. Since $\Omega(m)\neq o(\phi(m))$ there exists a $\delta>0$ such that $\Omega(m)\geqslant\delta\phi(m)$ for an infinite set $\mathbb{M}$ of positive integers m. Hence, using first part (ii) and then part (iii) of Lemma 2, we obtain, for all sufficiently large $m\in\mathbb{M}$,

$$
\begin{aligned}
A(m, \Omega) & \\
&\geqslant \frac{1}{p(u_m)} \sum_{m<v \leqslant m+\delta\phi(m)} u_m^v p_v \\
&= \sum_{m<v \leqslant m+\delta\phi(m)} \frac{p_v}{\Delta_v} \cdot \frac{p(u_v)\, u_m^v}{p(u_m)\, u_v^v} \\
&\geqslant \frac{1}{2} \sum_{m<v \leqslant m+\delta\phi(m)} \frac{p_v}{\Delta_v} \cdot \frac{\phi(v)}{\phi(m)} e^{g(m)-g(v)+(v-m)g'(m)} \\
&= \frac{1}{2} \sum_{m<v \leqslant m+\delta\phi(m)} \frac{p_v}{\Delta v} \cdot \frac{\phi(v)}{\phi(m)} e^{-(v-m)^2 g''(\xi)/2} \qquad (m \leqslant \xi \leqslant m+\delta\phi(m)) \\
&\geqslant \frac{1}{2} e^{-\delta^2 g''(\xi)/2g''(m)} \sum_{m<v \leqslant m+\delta\phi(m)} \frac{p_v}{\Delta_v} \\
&\geqslant \frac{\delta}{8\sqrt{2\pi}} e^{-\delta^2/2} > 0.
\end{aligned}
$$

Hence, by [9, Theorem 1], $\Omega(n)$ is not a summability function of the first kind for P, and thus also not a summability function of the first kind for J_p. Further, since a summability function of the second kind is also a summability function of the first kind for J_p, $\Omega(n)$ cannot be a summability function of the second kind for J_p either. ▮

3. Proofs of the Main Results

Proof of Theorem C (cf. [6, Theorem 2]). In view of (1.3) it suffices to prove that $s_n \to s(N, p, p)$, and because of regularity and linearity of this method we can suppose that $s=0$. Thus our hypothesis becomes

$$S_n := \sum_{k=0}^{n} s_k = o(\phi(n)).$$

Let $\varepsilon > 0$ be given. Then there is an $m \in \mathbb{N}$ such that $|S_n| < \varepsilon\phi(n)$ for $n \geqslant m$. By Abel partial summation we have, for $n \geqslant m$,

$$
\begin{aligned}
\sigma_n &:= \frac{1}{p_n^{*2}} \sum_{k=0}^{n} p_{n-k} p_k s_k \\
&= \frac{1}{p_n^{*2}} \sum_{k=0}^{n} S_k (p_{n-k} p_k - p_{n-k-1} p_{k+1}) \qquad (p_{-1} := 0) \\
&= \frac{1}{p_n^{*2}} \left(\sum_{k=0}^{m-1} + \sum_{k=m}^{n} \right) S_k (p_{n-k} p_k - p_{n-k-1} p_{k+1}) =: \Sigma_1 + \Sigma_2.
\end{aligned}
$$

Since m is fixed, it follows from the regularity of (N, p, p) that $\Sigma_1 \to 0$ as $n \to \infty$ [5, Theorem 2]. Next, we have, by first applying part (i) of Lemma 1 and then part (ii), that, as $n \to \infty$,

$$
\begin{aligned}
|\Sigma_2| &\leqslant \frac{\varepsilon\phi(n)}{p_n^{*2}} \sum_{k=0}^{n} |p_{n-k} p_k - p_{n-k-1} p_{k+1}| \\
&\leqslant \frac{\varepsilon\phi(n)}{p_n^{*2}} 2p_{n/2}^2 \sim \frac{2\varepsilon\phi(n)}{\sqrt{\pi}\,\phi(n/2)} \\
&= \frac{2\varepsilon n}{\sqrt{\pi}\,\phi(n/2)} \frac{\phi(n)}{n} \leqslant \frac{4\varepsilon}{\sqrt{\pi}},
\end{aligned}
$$

since $\phi(n)/n$ decreases. Hence $\limsup |\sigma_n| \leqslant 4\varepsilon/\sqrt{\pi}$, and therefore $\sigma_n \to 0$. ▮

Part (i) of Theorem 1 is an immediate consequence of Theorem C and the following lemma.

Lemma 4. *Suppose that the hypotheses of Theorem* 1 *hold. Then* $s_n^1 = s + o(\phi(n)/n)$.

Proof. We may suppose the $s=0$. Let

$$\tau_n := t_n Q_n, \qquad \text{so that} \quad \tau_n - \mu = o(\phi(n)\, q_n) \tag{3.1}$$

by (1.6).

Then, for $n \geqslant 1$,

$$s_n = \frac{\tau_n - \mu}{q_n} - \frac{\tau_{n-1} - \mu}{q_{n-1}} - (\tau_{n-1} - \mu)\left(\frac{1}{q_n} - \frac{1}{q_{n-1}}\right),$$

so that

$$\sum_{k=0}^{n} s_k = \frac{\tau_n - \mu}{q_n} - \sum_{k=1}^{n} (\tau_{k-1} - \mu)\left(\frac{1}{q_k} - \frac{1}{q_{k-1}}\right) + \frac{\mu}{q_0} \tag{3.2}$$

and so

$$s_n^1 = o\left(\frac{\phi(n)}{n}\right) - \frac{1}{n+1} \sum_{k=1}^{n} (\tau_{k-1} - \mu)\left(\frac{1}{q_k} - \frac{1}{q_{k-1}}\right). \tag{3.3}$$

Next, by (3.1), we have that $\tau_k = \mu + \varepsilon_k \phi(k)\, q_k$ where $\varepsilon_k \to 0$. Hence, by (1.4) and (1.5), we get that

$$\sum_{k=1}^{n} (\tau_{k-1}-\mu)\left(\frac{1}{q_k}-\frac{1}{q_{k-1}}\right)$$
$$=\sum_{k=1}^{n} \varepsilon_{k-1}\frac{\phi(k-1)}{k}\left(\frac{q_{k-1}}{q_k}-1\right)k=o(\phi(n)), \tag{3.4}$$

the final order relation being justified since $(q_{k-1}/q_k-1)k=O(1)$ and, for some $\varepsilon\in(0,1)$, $x^{-\varepsilon}\phi(x)$ increases for $x\geqslant x_0$, so that, for $n>x_0$,

$$\sum_{k=x_0}^{n}\frac{\phi(k-1)}{k}\leqslant\sum_{k=x_0}^{n}\frac{\phi(k)}{k}\leqslant n^{-\varepsilon}\phi(n)\sum_{k=x_0}^{n}k^{\varepsilon-1}=O(\phi(n)).$$

It follows from (3.3) and (3.4) that $s_n^1=o(\phi(n)/n)$. ∎

Proof of Theorem 1. As stated above part (i) follows from Lemma 4 and Theorem C. To prove parts (ii) and (iii) we observe that, by Lemma 3, there exists a bounded sequence (x_n) satisfying $x_n^1=O(\phi(n)/n)$ which is not J_p-summable. We now define (s_n) so that

$$\tau_n:=t_nQ_n=(n+1)\,q_nx_n^1.$$

Then

$$\tau_n=O(\phi(n)\,q_n). \tag{3.5}$$

It follows from (3.1) with $\mu=0$ that

$$s_n^1=x_n^1-y_n^1,\qquad \text{where } y_n:=\tau_{n-1}\left(\frac{1}{q_n}-\frac{1}{q_{n-1}}\right)\text{ for } n\geqslant 1 \text{ and } y_0:=0,$$

and so

$$s_n=x_n-y_n.$$

Next, by (3.5) and (1.5), we have that

$$y_n=\frac{\tau_{n-1}}{q_{n-1}}\left(\frac{q_{n-1}}{q_n}-1\right)=O\left(\frac{\phi(n)}{n}\right)=o(1),$$

so that (y_n) is J_p-summable. Since (x_n) is bounded and not J_p-summable, it follows that (s_n) also has these properties. In addition (s_n) is C_1-summable to 0, since $x_n^1=o(1)$. Finally, by (1.3), (s_n) is not summable by any member of Γ_p. ∎

Proof of Theorem 2. *For* $n\geqslant 1$, *let*

$$\psi(n):=\frac{n}{\phi(n)},\qquad \tau_n:=t_nQ_n,\qquad\text{and}\qquad z_n:=\tau_n-\mu\frac{\phi(n)}{n}.$$

Then, by hypothesis (1.10), we have that

$$\frac{1}{n}\sum_{k=1}^{n}\psi(k)\,z_k=o\left(\frac{\phi(n)}{n}\right), \tag{3.6}$$

from which it follows, by Theorem C, that

$$\psi(n)\,z_n\to 0(J_p). \tag{3.7}$$

Further, (3.6) implies that

$$z_n=o\left(\frac{\phi(n)^2}{n}\right)=o(1), \tag{3.8}$$

by (1.2) and (1.7). Next, using the notation $\Delta x_n:=x_n-x_{n-1}$, we have that

$$\Delta(\psi(n)\,z_n)=\psi(n)(z_n-z_{n-1})+z_{n-1}(\psi(n)-\psi(n-1))$$
$$=\psi(n)\left[\left(\tau_n-\mu\frac{\phi(n)}{n}\right)-\left(\tau_{n-1}-\mu\frac{\phi(n-1)}{n-1}\right)\right]$$
$$+z_{n-1}(\psi(n)-\psi(n-1))$$
$$=\psi(n)(\tau_n-\tau_{n-1})+\mu\psi(n)\left(\frac{\phi(n-1)}{n-1}-\frac{\phi(n)}{n}\right)$$
$$+z_{n-1}(\psi(n)-\psi(n-1)).$$

Hence

$$\phi(n)\,\Delta(\psi(n)\,z_n)=nq_ns_n+\mu n\left(\frac{\phi(n-1)}{n-1}-\frac{\phi(n)}{n}\right)$$
$$+z_{n-1}\phi(n)(\psi(n)-\psi(n-1)). \tag{3.9}$$

Now

$$0\leqslant n\left(\frac{\phi(n-1)}{n-1}-\frac{\phi(n)}{n}\right)\leqslant n\phi(n)\left(\frac{1}{n-1}-\frac{1}{n}\right)=\frac{\phi(n)}{n-1}=o(1), \tag{3.10}$$

and

$$0\leqslant\phi(n)(\psi(n)-\psi(n-1))$$
$$=\phi(n)\left(\frac{n}{\phi(n)}-\frac{n-1}{\phi(n-1)}\right)$$
$$\leqslant\frac{\phi(n)}{n}\frac{n}{\phi(n-1)}\leqslant\frac{n}{n-1}\leqslant 2. \tag{3.11}$$

Therefore, by (3.8), (3.9), (3.10), (3.11), and hypothesis (1.8), we have that

$$\Delta(\psi(n)\, z_n) = O_L\left(\frac{1}{\phi(n)}\right). \tag{3.12}$$

By virtue of a Tauberian theorem for J_p [4, see the remark after the corollary of Theorem 1], it follows from (3.7) and (3.12) that

$$\psi(n)\, z_n = o(1). \tag{3.13}$$

At this stage it is worth noting that if $\mu = 0$, then (3.13) and (1.9) imply that $\tau_n = o(\phi(n)\, q_n)$, so that the required conclusion follows from Theorem 1.

Returning to the general case, we deduce from (3.2) that

$$\begin{aligned} s_n^1 &= \frac{\tau_n}{(n+1)\, q_n} - \frac{1}{n+1} \sum_{k=1}^{n} \tau_{k-1} \left(\frac{1}{q_k} - \frac{1}{q_{k-1}}\right) \\ &= \frac{z_n + \mu\phi(n)/n}{(n+1)\, q_n} \\ &\quad - \frac{1}{n+1} \sum_{k=1}^{n-1} \left(z_k + \mu \frac{\phi(k)}{k}\right)\left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right) \\ &\quad - \frac{\tau_0}{n+1}\left(\frac{1}{q_1} - \frac{1}{q_0}\right). \end{aligned} \tag{3.14}$$

As in the proof of Lemma 4,

$$\frac{1}{\phi(n)} \sum_{k=1}^{n-1} \frac{\phi(k)}{k} = O(1),$$

and, by (1.5) and (1.7),

$$\frac{1}{q_{k+1}} - \frac{1}{q_k} = \left(\frac{q_k}{q_{k+1}} - 1\right) \frac{1}{q_k} = O(1).$$

Consequently it follows from (3.13) that

$$\frac{1}{\phi(n)} \sum_{k=1}^{n-1} \psi(k)\, z_k \psi(k)^{-1} \left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right) = o(1). \tag{3.15}$$

Next, by (3.3), (1.11), (3.14), and (3.15), we have that

$$s_n^1 + v_n^1 = o\left(\frac{\phi(n)}{n}\right),$$

where

$$v_n^1 := \frac{-\mu\phi(n)/n}{(n+1)\, q_n} + \frac{\mu}{n+1} \sum_{k=1}^{n-1} \frac{\phi(k)}{k} \left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right),$$

whence, for $n \geqslant 2$,

$$v_n = \frac{\mu}{q_n}\left(\frac{\phi(n-1)}{n-1} - \frac{\phi(n)}{n}\right) = o\left(\frac{1}{nq_n}\right) = o(1),$$

by (3.10) and (1.9). It now follows, by Theorem C and the regularity of the members of Γ_p, that (v_n) and $(s_n + v_n)$ are summable to 0 by every member of Γ_p, and therefore so also is (s_n). ∎

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