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On Methods of Summability Based on Power Series

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XXII.—On Methods of Summability Based on Power Series.*

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Synopsis

Given a power series $p(x)=\sum_0^\infty p_n x^n$ with real non-negative coefficients and having radius of convergence ρ, a summability method P is defined as follows:

$$s_n \to l \text{ (P) if } \frac{1}{p(x)} \sum_0^\infty p_n s_n x^n \to l \text{ as } x \to \rho - .$$

The main concern of this note is to establish conditions sufficient for one such method to include another.

1. Introduction

Suppose throughout that

$$p_n \geqslant 0, \qquad q_n \geqslant 0, \qquad \sum_{\nu=n}^\infty p_\nu > 0, \qquad \sum_{\nu=n}^\infty q_\nu > 0 \qquad (n=0, 1, \ldots).$$

Let

$$p(x)=\sum_{n=0}^\infty p_n x^n, \qquad q(x)=\sum_{n=0}^\infty q_n x^n$$

and denote the radii of convergence of the power series by ρ_p and ρ_q respectively.

Given any sequence $\{s_n\}$ of complex numbers we shall use the notations:

$$p_s(x)=\frac{1}{p(x)}\sum_{n=0}^\infty p_n s_n x^n, \qquad q_s(x)=\frac{1}{q(x)}\sum_{n=0}^\infty q_n s_n x^n.$$

If $\rho_p > 0$ and $\sum p_n s_n x^n$ is convergent in the open interval $(0, \rho_p)$, and if $p_s(x)$ tends to a finite limit l as $x \to \rho_p -$, we shall write

$$s_n \to l \text{ (P)}.$$

This defines the summability method P; the method Q, associated with the sequence $\{q_n\}$, is defined similarly. The method P is said to be regular if $s_n \to l$ (P) whenever $s_n \to l$. If $\rho_p = \rho_q > 0$ and $s_n \to l$ (P) whenever $s_n \to l$ (Q), P is said to include Q and we write $\mathrm{P} \supseteq \mathrm{Q}$. If $\mathrm{P} \supseteq \mathrm{Q}$ and $\mathrm{Q} \supseteq \mathrm{P}$ we write $\mathrm{P} \simeq \mathrm{Q}$.

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Suppose in what follows that χ is a real function of bounded variation in the interval $[0, 1]$. The main result to be established is:

Theorem *A*.—*If*

$$(a)\ p_n = q_n \int_0^1 t^n d\chi(t) \geqslant \delta q_n \int_0^1 t^n \,|\, d\chi(t) \,| \qquad (1 \geqslant \delta > 0,\ n=N, N+1, \ldots)$$

(b) $\rho_p = \rho_q > 0$ *and* P *is regular,*

then $\mathrm{P} \supseteq \mathrm{Q}$.

In the next section we prove three subsidiary theorems which have some bearing on Theorem *A*. Theorem 1 states necessary and sufficient conditions for P to be regular; it is not new but its proof has been included for the sake of completeness. Theorems 1 (ii) and 2 show that (*b*) is a consequence of (*a*) when $\rho_p = \infty$. Theorem 3 states a necessary and sufficient condition for ρ_p and ρ_q to be equal when $\infty > \rho_p > 0$ and (*a*) holds with a monotonic χ.

In § 4 Theorem *A* is linked with the theory of moment sequences; and in the final section some examples are given.

2. Subsidiary Theorems

Theorem* 1.—(i) *If* $\infty > \rho = \rho_p > 0$, *then a necessary and sufficient condition for* P *to be regular is that* $\sum p_n \rho^n = \infty$.

(ii) *If* $\rho_p = \infty$ *then* P *is regular.*

Proof of (i).—Note first that, since $p_n \geqslant 0$,

$$\sum_{n=0}^\infty p_n \rho^n \geqslant \lim_{x\to\rho-} p(x) \geqslant \sum_{n=0}^m p_n \rho^n \qquad (m \geqslant 0),$$

and hence that

$$\lim_{x\to\rho-} p(x) = \sum_{n=0}^\infty p_n \rho^n.$$

Sufficiency.—Suppose that $s_n \to 0$ and let m be any positive integer. Since, by hypothesis, $p(x) \to \infty$ as $x \to \rho -$, we have

$$\overline{\lim_{x\to\rho-}}\, |\, p_s(x) \,| \leqslant \overline{\lim_{x\to\rho-}} \frac{1}{p(x)} \sum_{n=m}^\infty p_n \,|\, s_n \,|\, x^n \leqslant \overline{\mathrm{bd}}_{n\geqslant m} \,|\, s_n \,|,$$

which tends to zero as $m \to \infty$. Hence $p_s(x) \to 0$ as $x \to \rho -$, and an immediate consequence is that P is regular.

* Hardy 1949, pp. 79–81.

Necessity.—Suppose that P is regular and let m be an integer such that $p_m > 0$. Define a sequence $\{s_n\}$ as follows:

$$s_n = 0 \quad (n \neq m), \qquad s_m = 1/p_m.$$

Then $s_n \to 0$ and therefore $p_s(x) = 1/p(x) \to 0$ as $x \to \rho -$. Hence

$$\sum_{n=0}^{\infty} p_n \rho^n = \lim_{x \to \rho -} p(x) = \infty.$$

Proof of (ii).—Suppose that $s_n \to 0$. Let m be any positive integer and let k be the first integer greater than m such that $p_k > 0$. Then

$$\overline{\lim_{x\to\infty}} \, | p_s(x) | \leq \overline{\lim_{x\to\infty}} \, \frac{x^m}{p_k x^k} \sum_{n=0}^{m} p_n | s_n | + \overline{\underset{n>m}{\mathrm{bd}}} \, | s_n | = \overline{\underset{n>m}{\mathrm{bd}}} \, | s_n |,$$

and the final expression tends to zero as $m \to \infty$. Hence P is regular.

THEOREM 2.—*If* $\quad p_n \geq \delta_n q_n \quad (n \geq N),$

where $$\infty > \delta_n = \int_0^1 t^n \, | d\chi(t) | > 0 \ (n \geq 0),$$

and if $\rho_p = \infty$, *then* $\rho_q = \infty$.

Proof.—Since $\delta_1 > 0$, $\int_u^1 | d\chi(t) |$ cannot be zero for all u in $(0, 1)$. Further

$$p_n \geq q_n u^n \int_u^1 | d\chi(t) | \qquad (1 > u > 0, \ n \geq N).$$

Since $\rho_p = \infty$, it follows that $\rho_q = \infty$.

THEOREM 3.—*Suppose that* $p_n = q_n \int_0^1 t^n d\chi(t)$ $(n \geq N)$ *where* χ *is non-decreasing and bounded in* $[0, 1]$, *and that* $\infty > \rho_p > 0$. *Then in order that* $\rho_p = \rho_q$ *it is necessary and sufficient that* $\chi(1) > \chi(t)$ *whenever* $1 > t > 0$.

Proof. Sufficiency.—Let u be any number in the open interval $(0, 1)$. Then $\int_u^1 d\chi(t) > 0$ and

$$q_n \int_u^1 d\chi(t) \geq p_n \geq q_n u^n \int_u^1 d\chi(t) \qquad (n \geq N).$$

Consequently $\rho_p \geq \rho_q \geq u\rho_p$, and it follows that $\rho_p = \rho_q$.

Necessity.—Suppose that u is such that $1 > u > 0$ and $\chi(1) = \chi(u)$. Then

$$p_n = q_n \int_0^u t^n d\chi(t) \leq q_n u^n \{\chi(1) - \chi(0)\} \qquad (n \geq N).$$

Since $\chi(1) > \chi(0)$ and $\rho_p = \rho_q$, it follows that $\rho_q \leq u\rho_p = u\rho_q$, which is only possible if $u = 1$, since $0 < \rho_q < \infty$. Hence the required result.

We shall use the following easily verified lemma in the proof of Theorem *A*.

LEMMA.—*If* P *is regular and* $p_n = q_n$ *for* $n > N$, *then* $\mathrm{P} \simeq \mathrm{Q}$.

3. PROOF OF THEOREM *A*

Suppose that condition (*a*) holds with $N = 0$. In view of the lemma this involves no loss in generality.

Let $\rho = \rho_p = \rho_q$ and let $\{s_n\}$ be any sequence such that $\sum q_n s_n w^n$ is convergent whenever $| w | < \rho$. Suppose that $0 < x < \rho$. The equality in (*a*) yields

$$\sum_{n=0}^{\infty} p_n s_n x^n = \sum_{n=0}^{\infty} q_n s_n x^n \int_0^1 t^n d\chi(t) = \int_0^1 d\chi(t) \sum_{n=0}^{\infty} q_n s_n (xt)^n;$$

the inversion being legitimate since

$$\int_0^1 | d\chi(t) | \sum_{n=0}^{\infty} q_n | s_n | (xt)^n \leq \int_0^1 | d\chi(t) | \sum_{n=0}^{\infty} q_n | s_n | x^n < \infty.$$

Hence

$$p_s(x) = \frac{1}{p(x)} \int_0^1 q_s(xt) q(xt) d\chi(t). \tag{1}$$

Similarly, using the inequality in (*a*), we obtain

$$p(x) \geq \delta \int_0^1 q(xt) \, | d\chi(t) |. \tag{2}$$

Further, in view of hypothesis (*b*) and Theorem 1,

$$p(x) \to \infty \quad \text{as} \quad x \to \rho -. \tag{3}$$

It follows from (1), (2) and (3) that, for $0 < w < \rho$,

$$\overline{\lim_{x\to\rho-}} \, | p_s(x) |$$

$$\leq \overline{\lim_{x\to\rho-}} \, \frac{1}{p(x)} \left| \int_0^{w/x} q_s(xt) q(xt) d\chi(t) \right| + \overline{\lim_{x\to\rho-}} \, \frac{1}{p(x)} \left| \int_{w/x}^1 q_s(xt) q(xt) d\chi(t) \right|$$

$$\leq \overline{\lim_{x\to\rho-}} \, \frac{1}{p(x)} \int_0^1 | d\chi(t) | \sum_{n=0}^{\infty} q_n | s_n | w^n + \delta^{-1} \, \overline{\underset{v>w}{\mathrm{bd}}} \, | q_s(v) | = \delta^{-1} \, \overline{\underset{v>w}{\mathrm{bd}}} \, | q_s(v) |.$$

Consequently $s_n \to 0$ (P) whenever $s_n \to 0$ (Q); and hence $\mathrm{P} \supseteq \mathrm{Q}$.

4. Moment Sequences

Given any function ϕ of bounded variation in [0, 1] we define an associated normalized function ϕ^* as follows:

$$\phi^*(t)=\begin{cases} 0 & (t=0)\\ \tfrac{1}{2}\{\phi(t+)+\phi(t-)\}-\phi(0) & (0<t<1)\\ \phi(1)-\phi(0) & (t=1).\end{cases}$$

Let α be a real function of bounded variation in [0, 1], then (Widder 1946, Theorem 8*a*)

$$\int_0^1 t^n d\alpha(t)=\int_0^1 t^n d\alpha^*(t) \qquad (n \geq 0),$$

and (Widder 1946, Theorem 8*b* and Hobson 1927, § 247)

$$\int_0^1 t^n \mid d\alpha(t)\mid \geq \int_0^1 t^n \mid d\alpha^*(t)\mid \qquad (n \geq 0).$$

Further, it is known (Hardy 1949, Theorem 203) that if β is a function of bounded variation in [0, 1] such that

$$\int_0^1 t^n d\alpha(t)=\int_0^1 t^n d\beta(t) \qquad (n=0, 1, \ldots),$$

then $\alpha^*(t)=\beta^*(t)$ for $0 \leq t \leq 1$.

A sequence $\{\mu_n\}$ is said to be an *m*-sequence (moment sequence) if

$$\mu_n=\int_0^1 t^n d\chi(t) \qquad (n \geq 0)$$

where χ is a real function of bounded variation in [0, 1]; if, in addition,

$$\mu_n \geq \delta\int_0^1 t^n \mid d\chi^*(t)\mid \qquad (1 \geq \delta > 0,\ n=N, N+1, \ldots),$$

we shall call $\{\mu_n\}$ an $\bar{m}$-sequence. In view of the introductory remarks in this section the definition of $\bar{m}$-sequences is unambiguous.

We can now re-word Theorem *A* as follows:

Theorem A'.—*If* $p_n=\mu_n q_n$ $(n \geq N)$, *where* $\{\mu_n\}$ *is an* $\bar{m}$*-sequence, and if* $p_\nu=p_q>0$ *and* P *is regular, then* $\mathrm{P} \supseteq \mathrm{Q}$.

We conclude this section with some useful results concerning $\bar{m}$-sequences.

A sequence $\{a_n\}$ is said to be totally monotone if

$$\Delta^k a_n=\sum_{\nu=0}^{k}(-1)^\nu\binom{k}{\nu}a_{n+\nu} \geq 0 \qquad (n=0, 1, \ldots;\ k=0, 1, \ldots).$$

It is known (Hardy 1949, 2, § 11.8) that a necessary and sufficient condition for $\{a_n\}$ to be totally monotone is that

$$a_n=\int_0^1 t^n dh(t) \qquad (n \geq 0),$$

where h is non-decreasing and bounded in [0, 1].

Hence $\{\mu_n\}$ is an *m*-sequence if and only if $\mu_n=a_n-b_n$, where $\{a_n\}$ and $\{b_n\}$ are totally monotone.

The following propositions are easily verified:

I. *If* $\{\mu_n\}$ *and* $\{\lambda_n\}$ *are* $\bar{m}$*-sequences, then so also are* $\{\mu_n\lambda_n\}$ *and* $\{\mu_n+\lambda_n\}$ (*cf.* Hardy 1949, Theorem 210).

II. *If* $\mu_n=a_n-b_n$ *where* $\{a_n\}$ *and* $\{b_n\}$ *are totally monotone, and if*

$$a_n \geq \gamma b_n \qquad (\gamma>1,\ n=N, N+1, \ldots),$$

then $\{\mu_n\}$ *is an* $\bar{m}$*-sequence.*

III. *Any m-sequence which converges to a positive limit is an* $\bar{m}$*-sequence.*

IV. *If both* $\{\mu_n\}$ *and* $\{1/\mu_n\}$ *are positive m-sequences, then they are* $\bar{m}$*-sequences.*

Note that III is a consequence of II, and IV a consequence of III.

5. Examples

Let

$$p_n^\alpha=\begin{cases}\dbinom{n+\alpha}{n} & (\alpha>-1)\\ (n+1)^{-1} & (\alpha=-1),\end{cases}$$

so that, for $|x|<1$,

$$\sum_{n=0}^{\infty} p_n^\alpha x^n=\begin{cases}(1-x)^{-\alpha-1} & (\alpha>-1)\\ -x\log(1-x) & (\alpha=-1).\end{cases}$$

Denote the power series summability method associated with the sequence $\{p_n^\alpha\}$ by A_α. A_0 is then the ordinary Abel method.

For $\alpha \geqslant -1$, the radius of convergence of $\sum p_n^\alpha x^n$ is unity and, by Theorem 1 (i), A_α is regular.

Let $\mu > \lambda > -1$. Then

$$\frac{p_n^\lambda}{p_n^\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda)\Gamma(\lambda+1)} \int_0^1 t^n . t^\lambda (1-t)^{\mu-\lambda-1} dt,$$

so that the sequence $\{p_n^\lambda / p_n^\mu\}$ is totally monotone. Further

$$\frac{p_n^{-1}}{p_n^\mu} = \frac{1}{\mu+1} \cdot \frac{n+\mu+1}{n+1} \cdot \frac{1}{p_n^{\mu+1}}:$$

since $\{1/p_n^{\mu+1}\}$ is totally monotone and $\{(n+\mu+1)/(n+1)\}$ is an m-sequence (Hardy 1949, 264) which converges to unity, it follows, in view of propositions I and III, that $\{p_n^{-1}/p_n^\mu\}$ is an $\bar{m}$-sequence.

Hence, by Theorem *A*′,

$$A_\lambda \supseteq A_\mu \qquad (\mu > \lambda \geqslant -1).*$$

Denote by A_α' the power series method associated with the sequence $\{(n+1)^\alpha\}$. It is known that, for $\alpha \geqslant -1$, both $\{p_n^\alpha/(n+1)^\alpha\}$ and $\{(n+1)^\alpha / p_n^\alpha\}$ are m-sequences (Hardy 1949, Theorem 211). Hence, by proposition IV and Theorem *A*′,

$$A_\alpha' \simeq A_\alpha \qquad (\alpha \geqslant -1).$$

Let

$$q_0^\alpha = 1,\ q_n^\alpha = \frac{1}{(\alpha+1)(\alpha+2)\ldots(\alpha+n)} \qquad (\alpha > -1,\ n = 1, 2, \ldots),$$

and denote by B_α the method associated with the sequence $\{q_n^\alpha\}$. B_0 is then the Borel exponential method.

The series $\sum q_n^\alpha x^n$ is convergent for all x and hence, by Theorem 1 (ii), B_α is regular for $\alpha > -1$. Since

$$\frac{q_n^\mu}{q_n^\lambda} = \frac{p_n^\lambda}{p_n^\mu} \qquad (\lambda > -1,\ \mu > -1),$$

it follows that

$$B_\mu \supseteq B_\lambda \qquad (\mu > \lambda > -1).$$

Finally, denote by B_α' the method associated with the sequence $\{1/(n+1)^\alpha n!\}$, where α is any real number. As before we obtain

$$B_\alpha' \simeq B_\alpha \qquad (\alpha > -1),$$

from which it follows that $B_\mu' \supseteq B_\lambda'$ when $\mu > \lambda$ and $\lambda > -1$. However the restriction $\lambda > -1$ is unnecessary in this case. Since B_α' is regular for all real α and the sequence $\{(n+1)^{\lambda-\mu}\}$ is totally monotone whenever $\mu - \lambda > 0$ (Hardy 1949, 266), we have, by Theorem *A*′,

$$B_\mu' \supseteq B_\lambda' \qquad (\mu > \lambda).$$

* See Borwein 1957, where the result $A_\lambda \supset A_\mu$ $(\mu > \lambda > -1)$ is proved.

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