Some characterizations of Lindenstrauss spaces whose duals lack the weak* fixed point property for nonexpansive mappings

Łukasz Piasecki
joint work with Emanuele Casini and Enrico Miglierina

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Definition

Let X be an infinite dimensional real Banach space. We say that a nonempty bounded closed and convex subset C of X has the fixed point property (shortly, FPP) if each nonexpansive mapping (i.e., the mapping $T : C \to C$ such that $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$) has a fixed point. The space X^* is said to have the $\sigma(X^*, X)$-fixed point property ($\sigma(X^*, X)$-FPP) if every nonempty, convex, w^*-compact subset C of X^* has the FPP.

A Banach space X is called an L_1-*predual* or a *Lindenstrauss space* if its dual is isometric to an $L_1(\mu)$ for some measure μ.
This class of spaces was studied by J. Lindenstrauss, A. J. Lazar, E. Michael, A. Pełczyński, M. Zippin, D. E. Alspach, W. B. Johnson, Z. Semadeni, A. Gleit, R. McGuigan, Y. Benyamini, H. E. Lacey, W. Lusky, D. Wulbert, I. Gasparis, V. P. Fonf, P. Wojtaszczyk, and others...

It is well-known (L. A. Karlovitz, 1976) that ℓ_1 has the $\sigma(\ell_1, c_0)$-FPP whereas it lacks the $\sigma(\ell_1, c)$-FPP. Moreover, $C(K)^*$ fails w*-FPP (M. Smyth, 1995).

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Let X be a separable Banach space that contains an isometric copy of c. Then there is a subspace Y of X such that Y is isometric to c and 1-complemented in X.

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Remark

C. Lennard found an example of a convex, w^*-compact set $C \subset c^*$ that fails the FPP for affine (as well as for non affine) contractive mappings. Therefore, under the same assumptions of the previous theorem, X^* fails $\sigma(X^*, X)$-FPP for affine contractive mappings.

Corollary

Let X be a separable Lindenstrauss space such that X^* is a non-separable space. Then X^* lacks the $\sigma(X^*, X)$-FPP.

Proof.

Lazar and Lindenstrauss (1971) proved that a separable Lindenstrauss space X with nonseparable dual contains a subspace isometric to the space $\mathcal{C}(\Delta)$ where Δ is the Cantor set. Since $\mathcal{C}(\Delta)$ contains an isometric copy of c, the thesis follows directly from our Theorem. □

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Remark

Let X be a separable Banach space and let us suppose that there exists a quotient X/Y of X that contains an isometric copy of c. From our Theorem we know that $Y^\perp$ fails the w^*-FPP and it follows easily that also X^* fails the w^*-FPP.

Let $f=(f(1),f(2),\dots)\in\ell_1=c^*$ be such that $\|f\|=1$. We consider the hyperplane of c defined by

$$W_f=\{x\in c : f(x)=0\}=\left\{x\in c : \sum_{n=0}^{\infty} f(n+1)x(n)=0\right\}.$$

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$$W_f = \{x \in c : f(x) = 0\} = \left\{x \in c : \sum_{n=0}^{\infty} f(n+1)x(n) = 0\right\}.$$

Proposition (E. Casini, E. Miglierina, ŁP)

- W_f^* is isometric to ℓ_1 iff there exists $j_0 \geq 1$ such that $|f(j_0)| \geq 1/2$.
- W_f is isometric to c iff there exists $j_0 \geq 2$ such that $|f(j_0)| \geq 1/2$.
- W_f is isometric to c_0 iff $|f(1)| = 1$.

The hyperplanes W_f of c such that $W_f^* = \ell_1$ can be divided into three distinct classes:

- $W_f \simeq c$ (or, equivalently, there exists $j_0 \geq 2$ such that $|f(j_0)| \geq \frac{1}{2}$);
- W_f is isometric neither to c nor to c_0 (or, equivalently, $\frac{1}{2} \leq |f(1)| < 1$ and $|f(j)| < \frac{1}{2}$ for every $j \geq 2$);
- $W_f \simeq c_0$ (or, equivalently, $|f(1)| = 1$).

Proposition (E. Casini, E. Miglierina, ŁP)

Let $W_f \subset c$ be such that $\frac{1}{2} \leq |f(1)| \leq 1$ and $|f(j)| < \frac{1}{2}$ for every $j \geq 2$. If $\{e_n^*\}$ is the standard basis of ℓ_1, then

$$e_n^* \overset{\sigma(\ell_1, W_f)}{\longrightarrow} \hat{e},$$

where $\hat{e} = \left(-\frac{f(2)}{f(1)}, -\frac{f(3)}{f(1)}, -\frac{f(4)}{f(1)}, \dots\right)$.

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Corollary (E. Casini, E. Miglierina, ŁP)

Let X be a Banach space such that $X^* = \ell_1$. If the standard basis $\{e_n^*\}$ of ℓ_1 is a $\sigma(\ell_1, X)$-convergent sequence, then there exists $f \in \ell_1$ with $\|f\|_{\ell_1} = 1$ such that X is isometric to W_f.

Theorem (M. A. Japón-Pineda, S. Prus)

Let τ be a locally convex topology in the real space ℓ_1 coarser than the weak topology on the unit ball. Assume that (e_n) converges to some $e \in \ell_1$ with respect to τ. Then ℓ_1 has the τ-FPP if and only if one of the following conditions holds

1. $\|e\| < 1$
2. $\|e\| = 1$ *and the set* $N^+ = \{n \in \mathbb{N} : e(n) \geq 0\}$ *is finite.*

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Definition (E. Casini, E. Miglierina, ŁP)

A space W_f is called "bad with respect to w^*-FPP" (shortly "bad") if $f \in \ell_1$ is such that $\|f\| = 1$, $|f(1)| = \frac{1}{2}$ and the set $N^+ = \{n \in \mathbb{N} : f(1)f(n+1) \leq 0\}$ is infinite.

Example (E. Casini, E. Miglierina, ŁP)

Let us consider the space W_f where

$$f = \left(-\frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{8}, 0, \frac{1}{16}, 0, \dots\right) \in \ell_1.$$

We have that

- $W_f^* = \ell_1$;
- W_f does not contain an isometric copy of c;
- there exists a quotient of W_f isometric to c;
- ℓ_1 lacks the $\sigma(\ell_1, W_f)$-FPP.

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We have that

- $W_f^* = \ell_1$;
- ℓ_1 lacks the $\sigma(\ell_1, W_f)$-FPP;
- W_f does not contain an isometric copy of c;
- W_f does not have a quotient that contains an isometric copy of c.

Theorem (E. Casini, E. Miglierina, ŁP)

Let X be a separable Banach space. If X contains a subspace isometric to a "bad" W_f, then X^* fails the $\sigma(X^*, X)$-FPP.

Remark

Let X be a separable Banach space and let us suppose that a "bad" W_f is a subspace of a quotient X/Y of X. Then X^* fails the w^*-FPP.

Theorem (E. Casini, E. Miglierina, ŁP)

Let X be a predual of ℓ_1. Then the following are equivalent.

1. ℓ_1 *lacks the* $\sigma(\ell_1, X)$*-FPP for nonexpansive mappings.*
2. ℓ_1 *lacks the* $\sigma(\ell_1, X)$*-FPP for isometries.*
3. ℓ_1 *lacks the* $\sigma(\ell_1, X)$*-FPP for contractive mappings.*
4. *There is a subsequence* $(e^*_{n_k})_{k\in\mathbb{N}}$ *of the standard basis* $(e^*_n)_{n\in\mathbb{N}}$ *in* ℓ_1 *which is* $\sigma(\ell_1, X)$*-convergent to a norm-one element* $e^* \in \ell_1$ *with* $e^*(n_k) \geq 0$ *for all* $k \in \mathbb{N}$*.*
5. *There is a quotient of* X *isometric to a "bad"* W_f*.*
6. *There is a quotient of* X *that contains a subspace isometric to a "bad"* W_g*.*

Remark

The spaces "bad" W_f and W_g in the statements (5) and (6) cannot be replaced by the space c.