Complete Lyapunov Functions and Control

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Outline

- Lyapunov's Second Method
- Control Lyapunov Functions (CLFs)
- Fundamental Difficulties
- Complete Lyapunov Functions and Perplexity

Lyapunov's Second Method

Theorem: Given $\dot{x} = f(x)$ with $f(0) = 0$. If there exists a continuously differentiable $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is positive definite and radially unbounded and satisfies

$$\frac{d}{dt} V(x(t)) = \langle \nabla V(x), f(x) \rangle = L_f V(x) < 0$$

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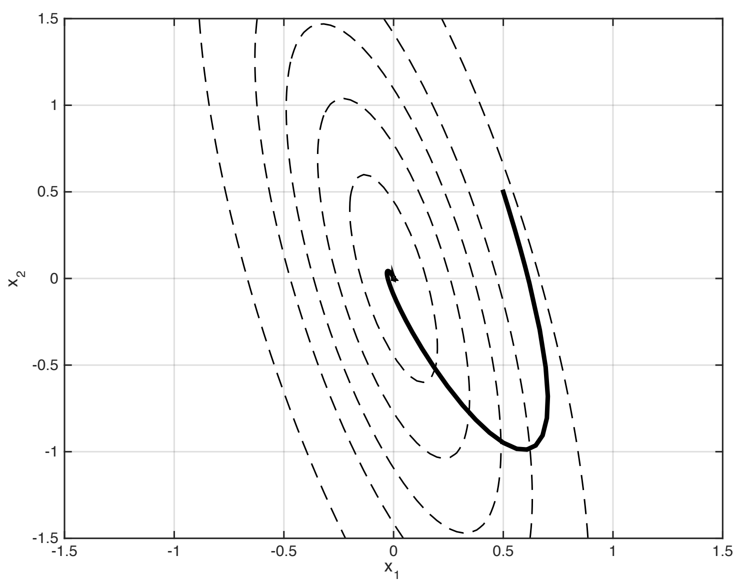
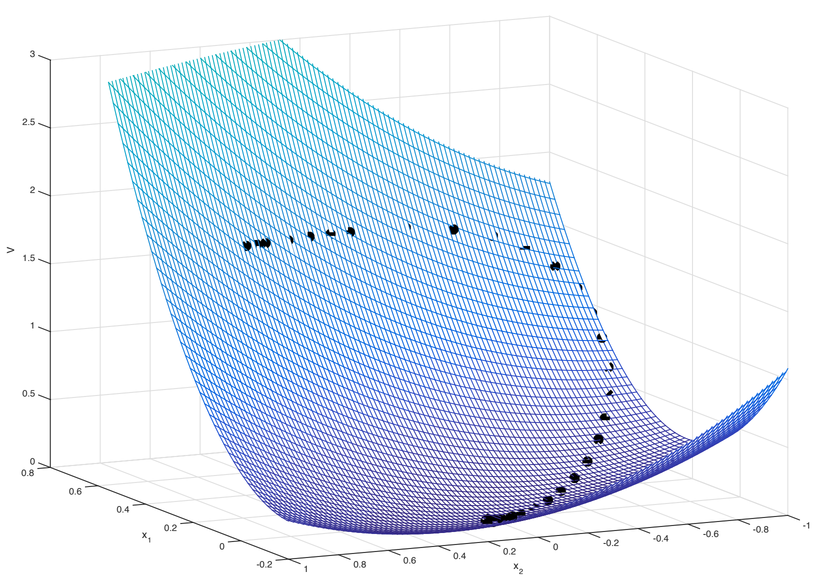
A Lyapunov function for $\dot{x} = \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix} x$ is $V(x) = x^T \begin{bmatrix} 4.5 & 1 \\ 1 & 0.5 \end{bmatrix} x$.





Jurdjevic-Quinn (Nonlinear Damping)

Definition: A control Lyapunov function for $\dot{x} = f(x) + g(x)u$ is a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$L_g V(x) = 0 \quad \Rightarrow \quad L_f V(x) < 0 \ \text{ for } \ x \neq 0.$$

Idea: $\frac{d}{dt} V(x(t)) = \langle \nabla V(x), f(x) + g(x)u \rangle = L_f V(x) + L_g V(x) u$

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Example: $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_1 u \quad$ and $\quad V(x) = \frac{1}{2}(x_1^2 + x_2^2).$

$$\dot{V} = x_1 x_2 - x_1 x_2 + x_1 x_2 u \quad \Rightarrow \quad u = -L_g V(x) = -x_1 x_2$$

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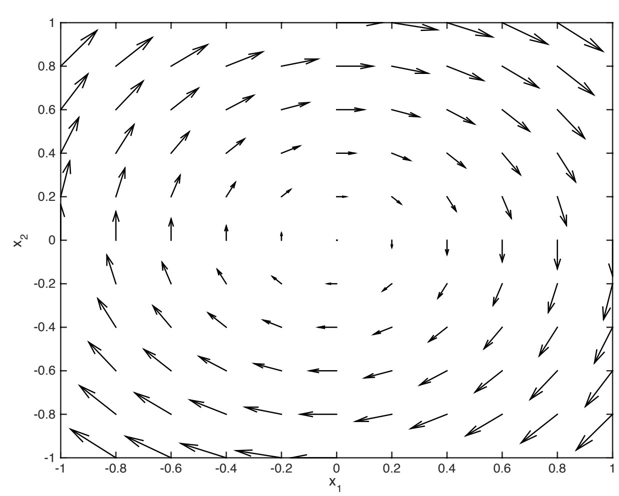

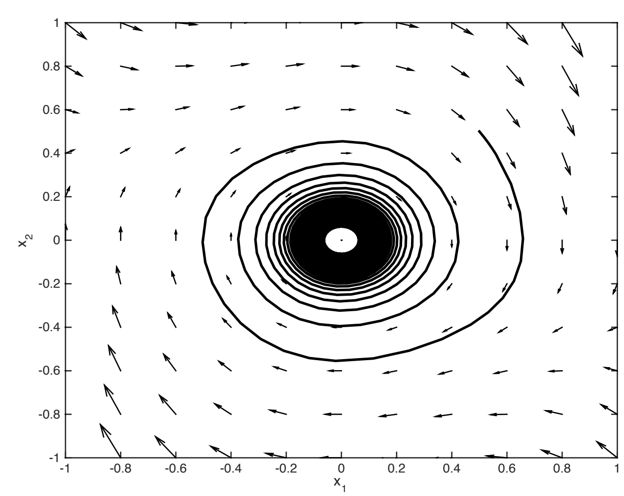

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Sontag's Formula:

$$u(x) := \begin{cases} -\left(\frac{L_f V(x) + \sqrt{L_f V(x)^2 + L_g V(x)^2}}{L_g V(x)^2}\right) L_g V(x) & , \quad L_g V(x) \neq 0 \\ 0 & , \quad L_g V(x) = 0 \end{cases}$$

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Idea:

$$\frac{d}{dt} V(x(t)) = L_f V(x) + L_g V(x) u = L_f V(x) - \left(\frac{L_f V(x) + \sqrt{L_f V(x)^2 + L_g V(x)^2}}{L_g V(x)^2}\right) L_g V(x)^2$$

$$= -\sqrt{L_f V(x)^2 + L_g V(x)^2} < 0$$

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Differential Inclusions - Controllability

System with input: $\dot{x} = f(x, u)$, $x \in \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^m$

Differential inclusion: $\dot{x} \in F(x) := \mathrm{co}\left(\bigcup_{u \in \mathcal{U}} f(x, u)\right)$

Comparison Functions: continuous $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

- Class-$\mathcal{K}_\infty$: zero at zero, strictly increasing, unbounded.
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$\mathcal{KL}$-stability: there exists $\beta \in \mathcal{KL}$ so that $|\phi(t,x)| \leq \beta(|x|,t), \quad \forall x \in \mathbb{R}^n,\ t \in \mathbb{R}_{\geq 0}$.

Strong $\mathcal{KL}$-stability: All solutions $\phi \in \mathcal{S}(x)$
Weak $\mathcal{KL}$-stability: At least one solution $\phi \in \mathcal{S}(x)$

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Uniform Global Asymptotic Controllability: There exists $\beta \in \mathcal{KL}$ so that, for each $x \in \mathbb{R}^n$ there exists $u \in \mathcal{U}$ so that

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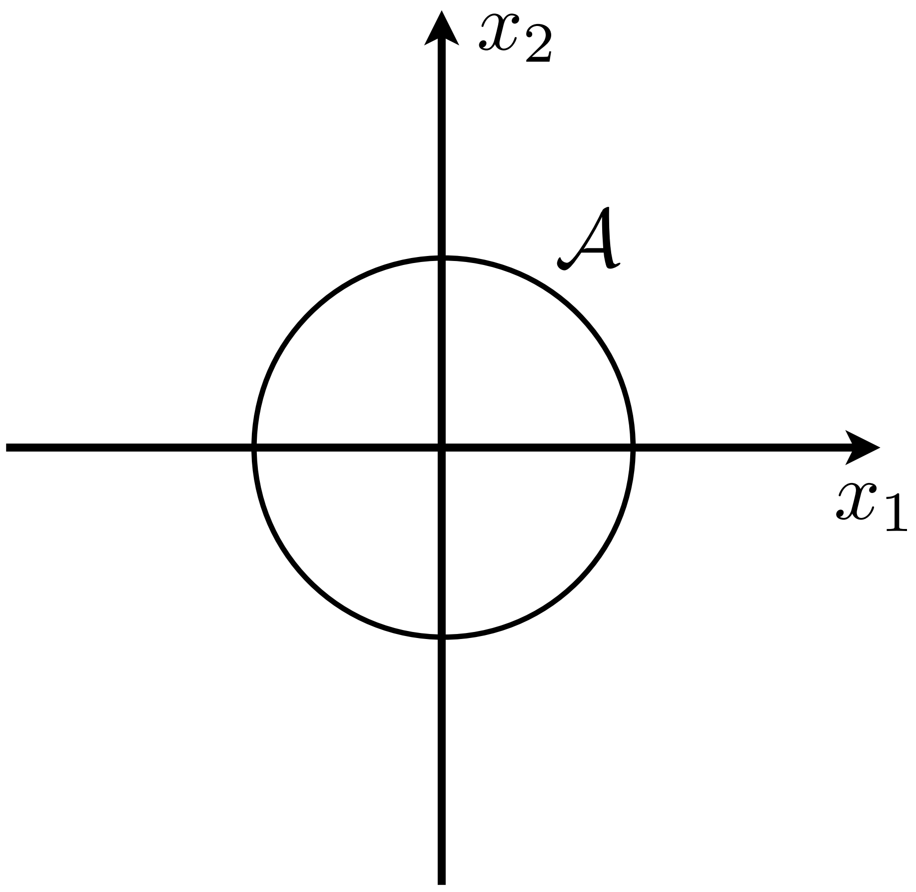
Smooth Lyapunov Function?

Consider $\dot{x} \in \overline{\mathcal{B}}$, $x \in \mathbb{R}^2$. The set $\mathcal{A} := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\right\}$ is weakly $\mathcal{KL}$-stable.



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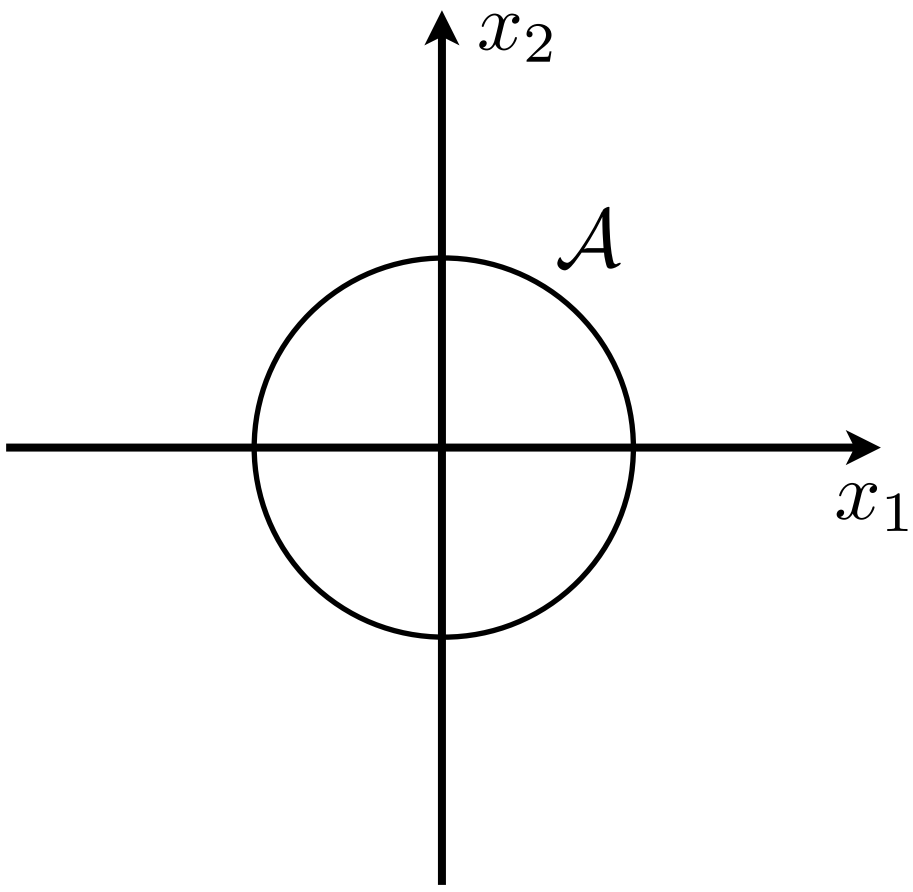

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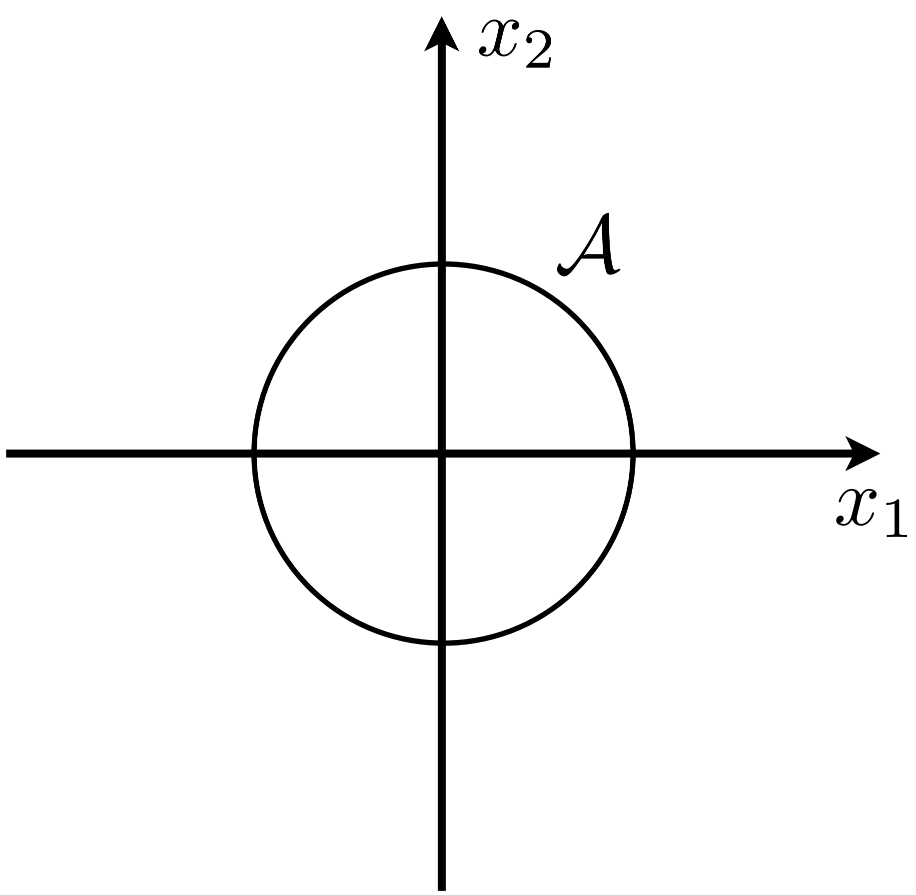

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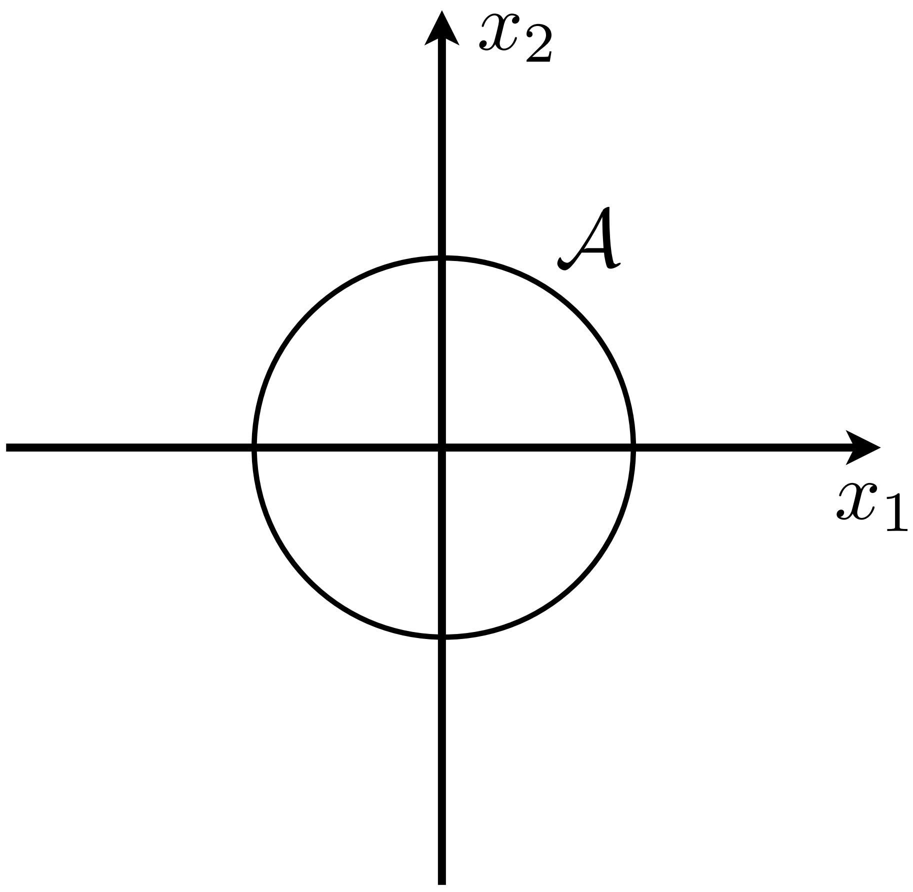

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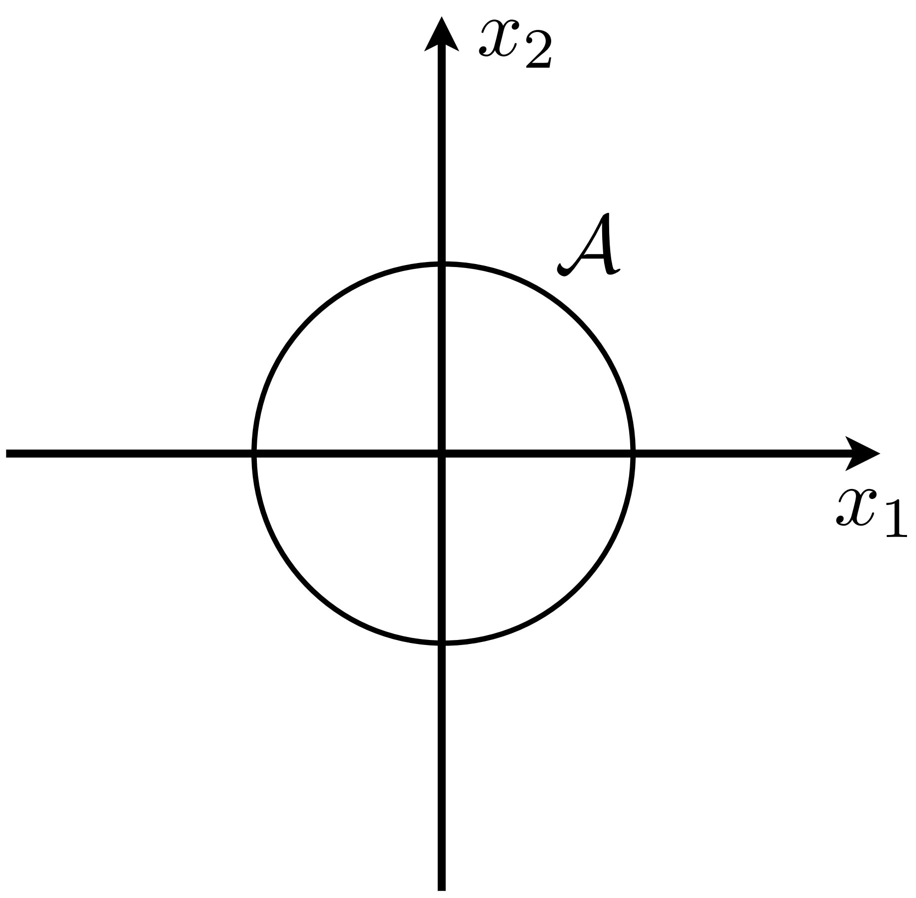

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Note: V attains maximum in interior of $\overline{\mathcal{B}}$.

Covering Condition

Theorem: Suppose $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies certain basic conditions (e.g., convex) and there exists a continuously differentiable weak Lyapunov function; i.e., a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathbb{R}^n$

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Then, for any $\gamma \in \mathbb{R}_{>0}$ there exists $\Delta \in \mathbb{R}_{>0}$ such that

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Brockett's (Nonholonomic) Integrator

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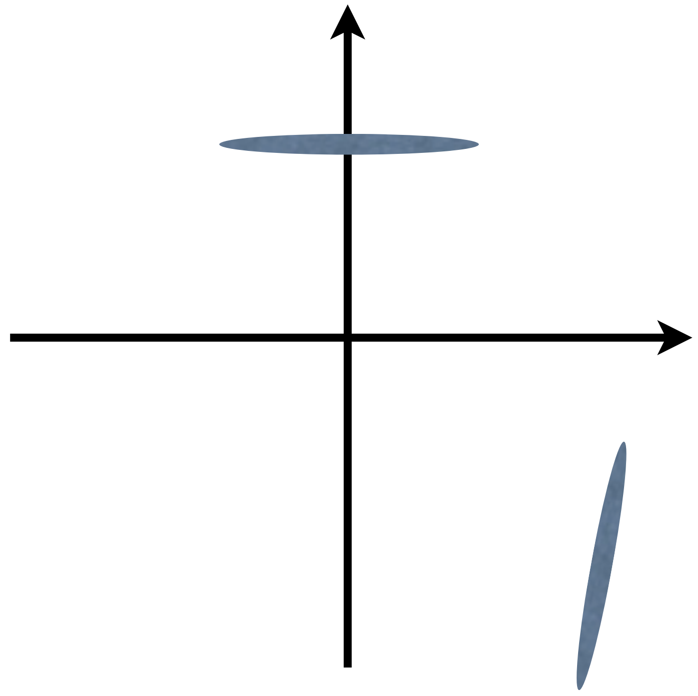

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Another Example

Brockett's Condition: Continuous feedback stabilizer implies for every $\gamma \in \mathbb{R}_{>0}$ there exists $\Delta \in \mathbb{R}_{>0}$ such that

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Does not satisfy Brockett's condition $\Rightarrow$ no continuous feedback stabiliser.

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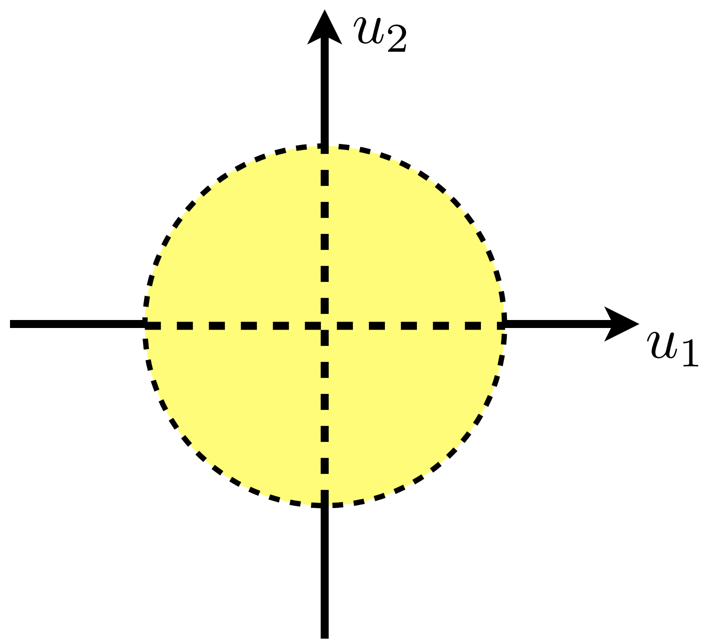

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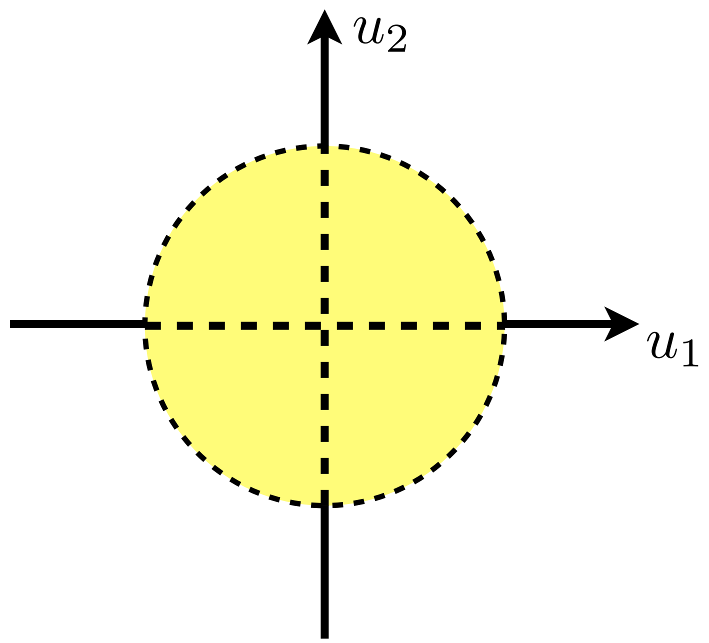
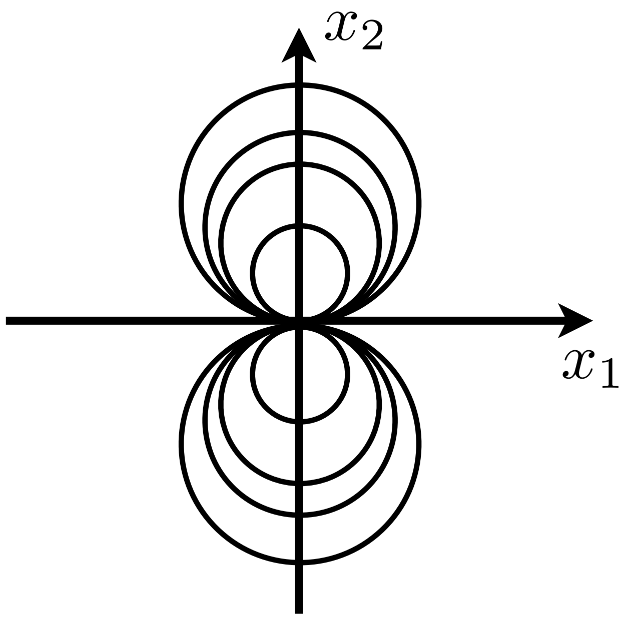


To deal with the most general cases, we will need to resort to nonsmooth CLFs and discontinuous feedbacks.

Artstein's Circles



$$\dot{x}_1 = (x_1^2 - x_2^2)u$$
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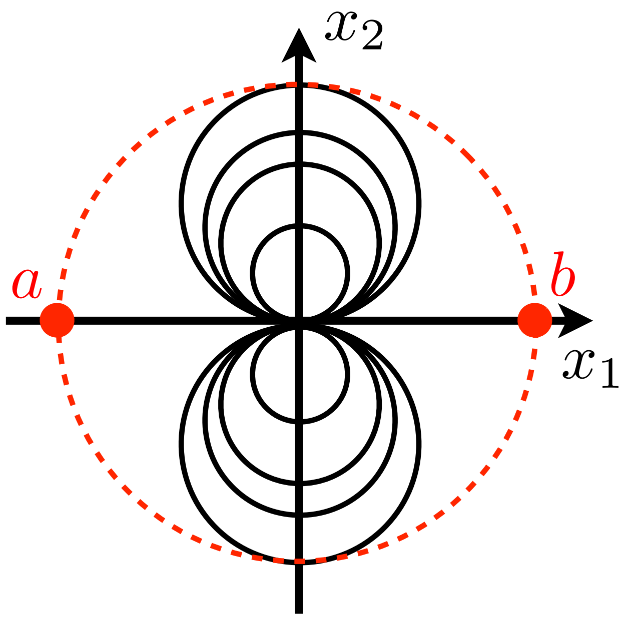

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Continuous stabilizer then requires $u = 0$ *somewhere* $\Rightarrow$ equilibrium.

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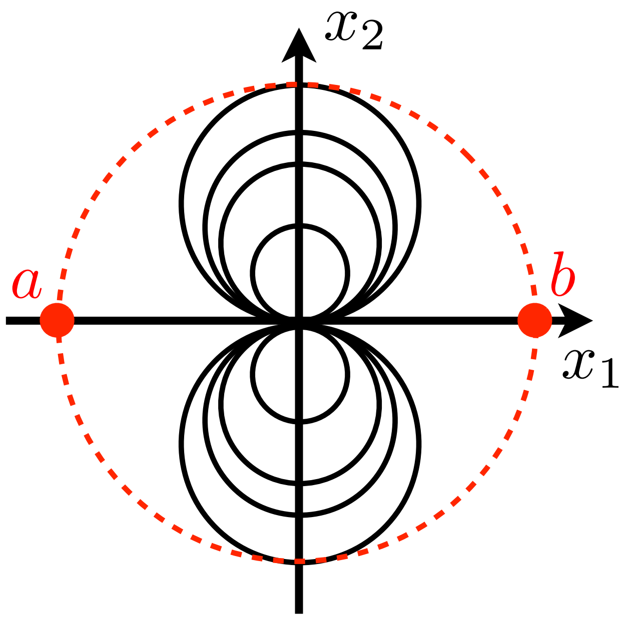

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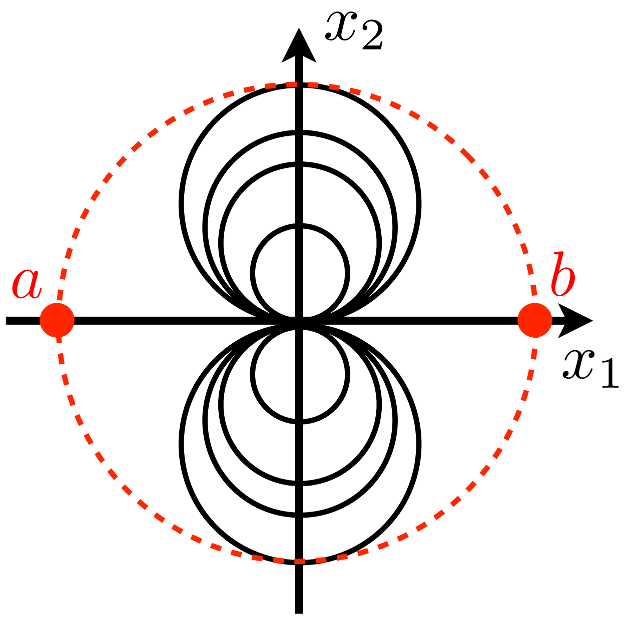

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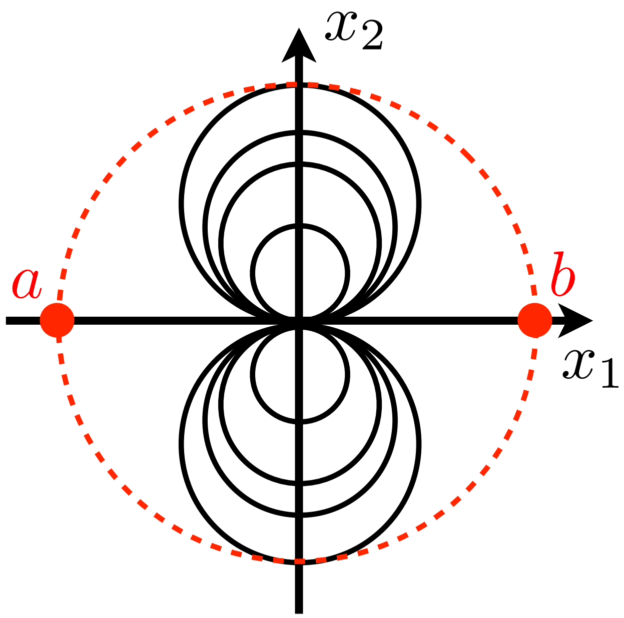
Discontinuous stabilizer and robustness for $u = \kappa(x + d)$? Solutions?

Artstein's Circles



$$\dot{x}_1 = (x_1^2 - x_2^2)u$$
$$\dot{x}_2 = 2x_1 x_2 u$$

$u > 0 \quad \Rightarrow \quad$ counterclockwise

$u < 0 \quad \Rightarrow \quad$ clockwise (reversed for lower circles)

Near the origin, stability requires $u > 0$ to the left and $u < 0$ to the right.

Continuous stabilizer then requires $u = 0$ *somewhere* $\Rightarrow$ equilibrium.

Control-affine: If there exists a differentiable CLF, then Sontag's formula is a continuous feedback stabilizer.

No continuous feedback stabilizer implies no differentiable CLF!

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Solution: Sample-and-hold (i.e., discrete time).

Nonsmooth Control Lyapunov Functions

Lower Dini Derivative:

$$DV(x;w) := \liminf_{\xi\to w,\varepsilon\to 0^+} \frac{V(x+\varepsilon\xi)-V(x)}{\varepsilon} = \liminf_{\varepsilon\to 0^+} \frac{V(x+\varepsilon w)-V(x)}{\varepsilon}$$

Definition: For $\dot{x} = f(x,u)$, $x \in \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^m$, a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a nonsmooth CLF if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that, for every $x \in \mathbb{R}^n$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \text{and}$$

$$\min_{w\in f(x,\mathcal{U})} DV(x;w) < 0.$$

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Similar definition with proximal subgradients: $\sup_{\zeta \in \partial_P V(x)} \min_{u \in \mathcal{U}} \langle \zeta, f(x,w) \rangle < 0.$

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(Discontinuous) Control:

1. In $\mathcal{V}(\ell_1, \ell_2)$, fix $r \in \left(0, \min\left\{\frac{\varepsilon_2}{L_V}, \varepsilon_3, \varepsilon_4, \frac{c}{L_f L_V}\right\}\right]$.
2. Measure x. For each $x \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2)$,
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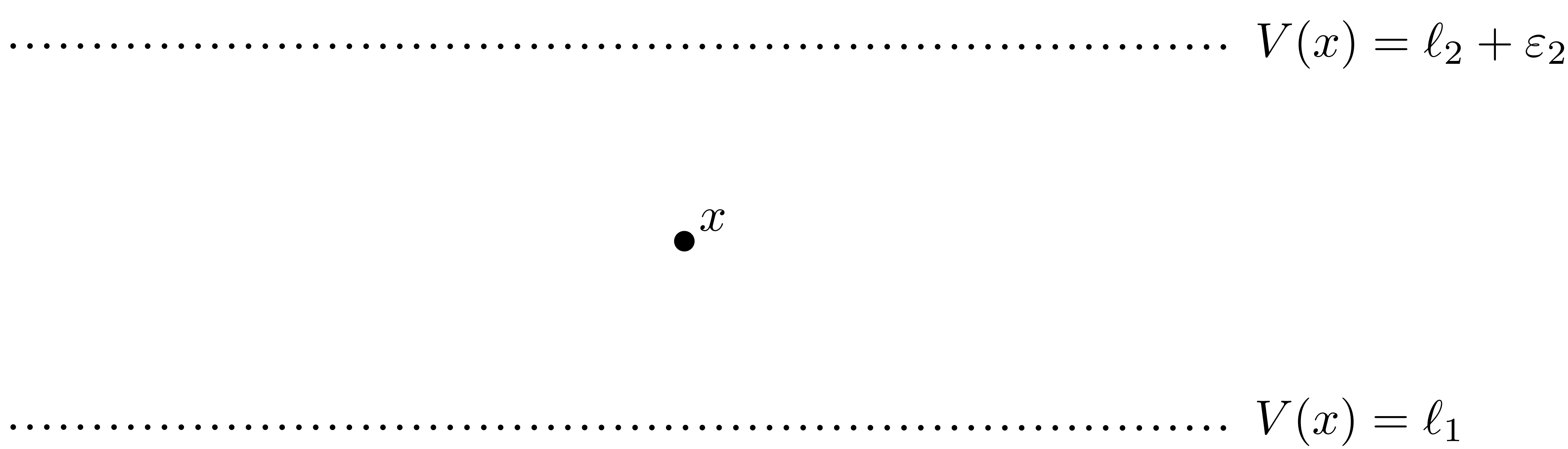

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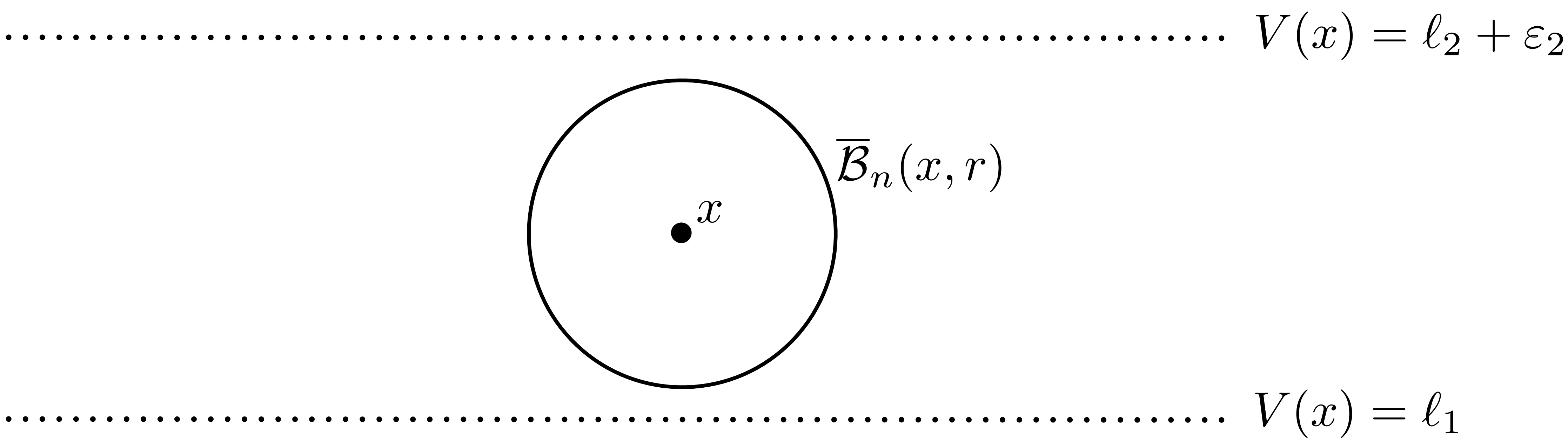

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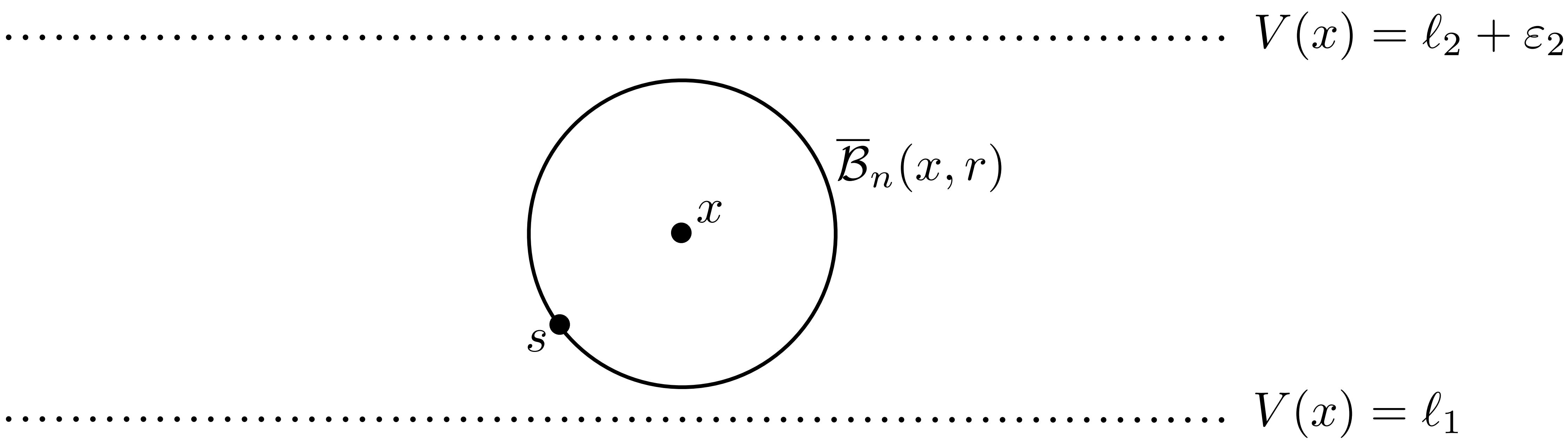

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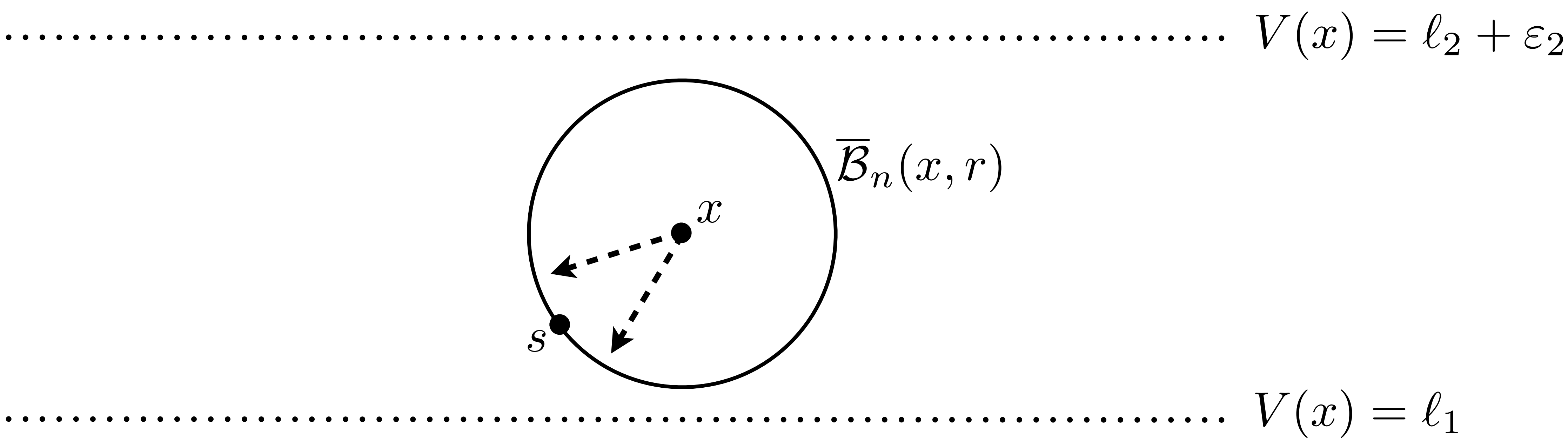

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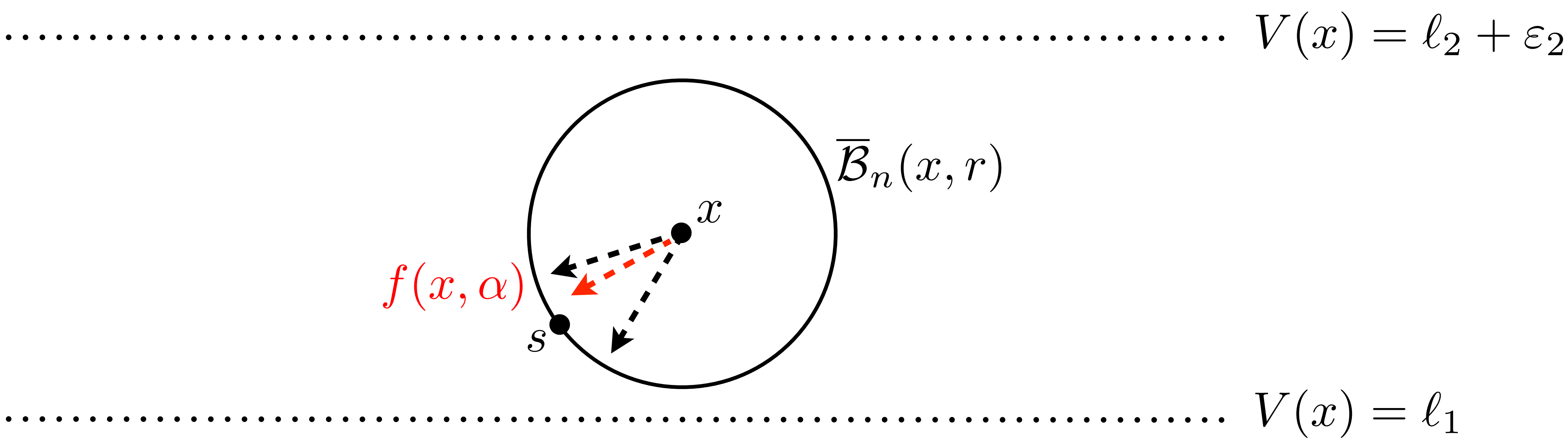

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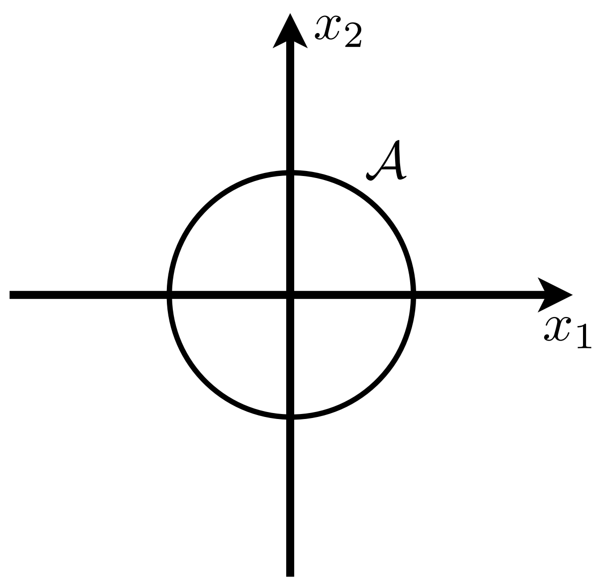
Instability and Constraints

Goal: For $\dot{x} = u$, $x \in \mathbb{R}^2$, $u \in [-1, 1]^2$, stabilise $\mathcal{A} := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\right\}$.



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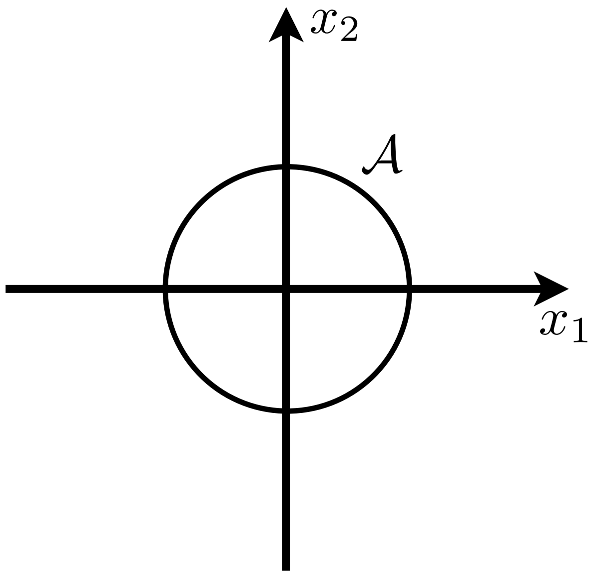

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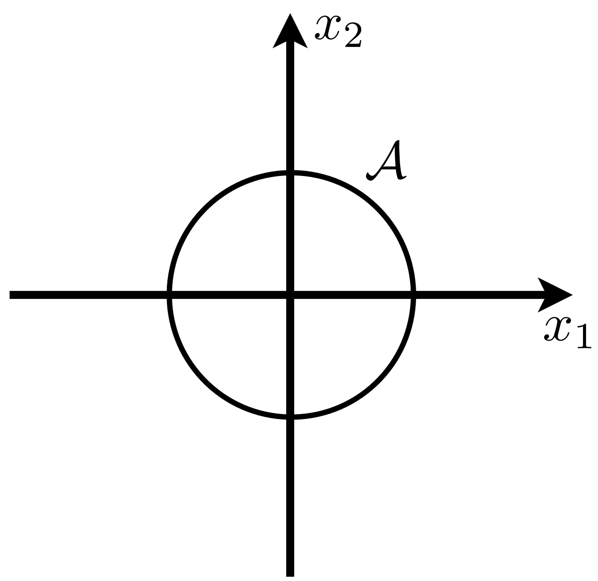

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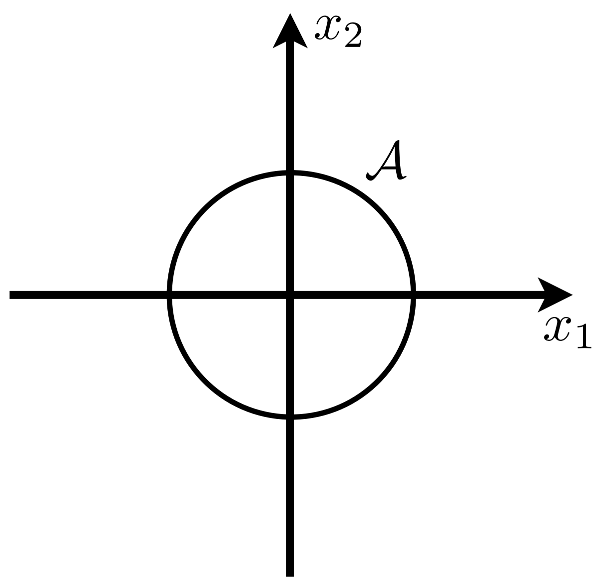

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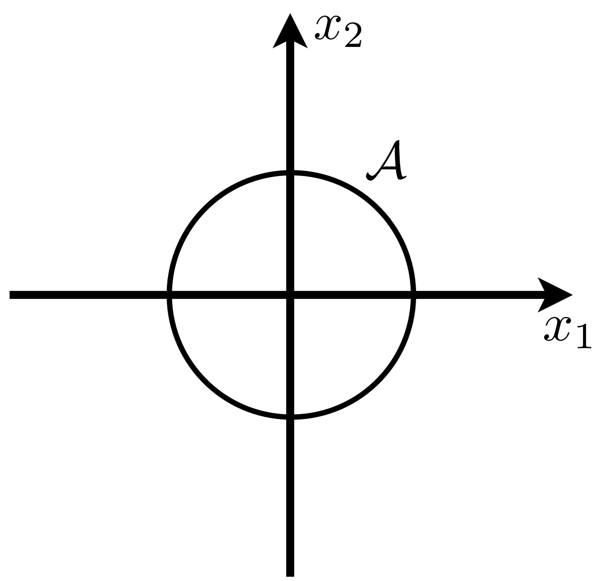

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Idea: Patch together stabilising / destabilising controllers (e.g., via hysteresis).

Example

Consider $\dot{x} = f(x) + g(x)u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

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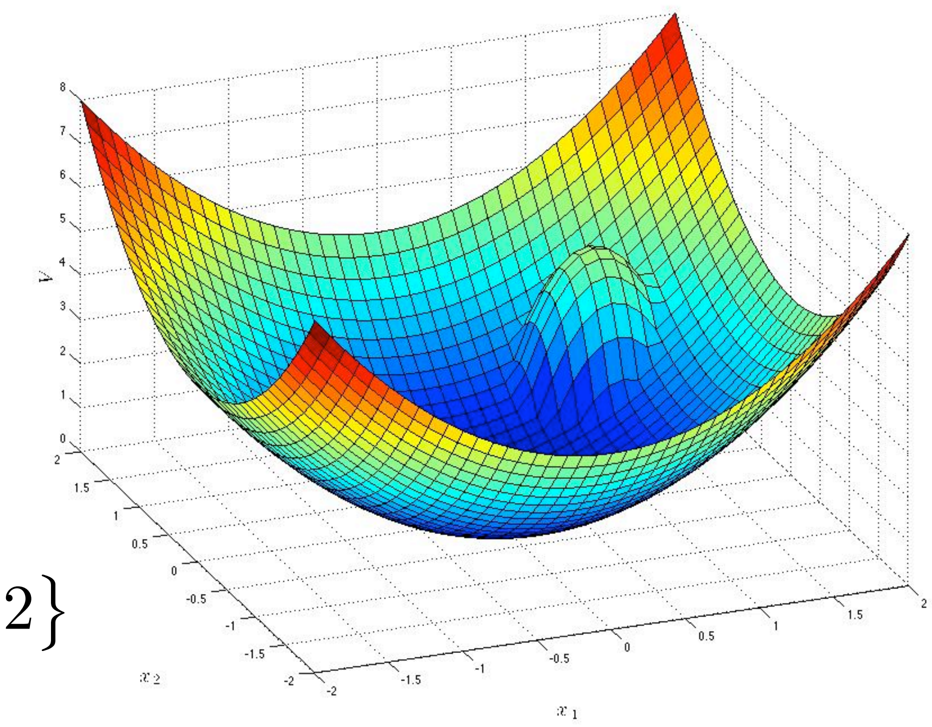

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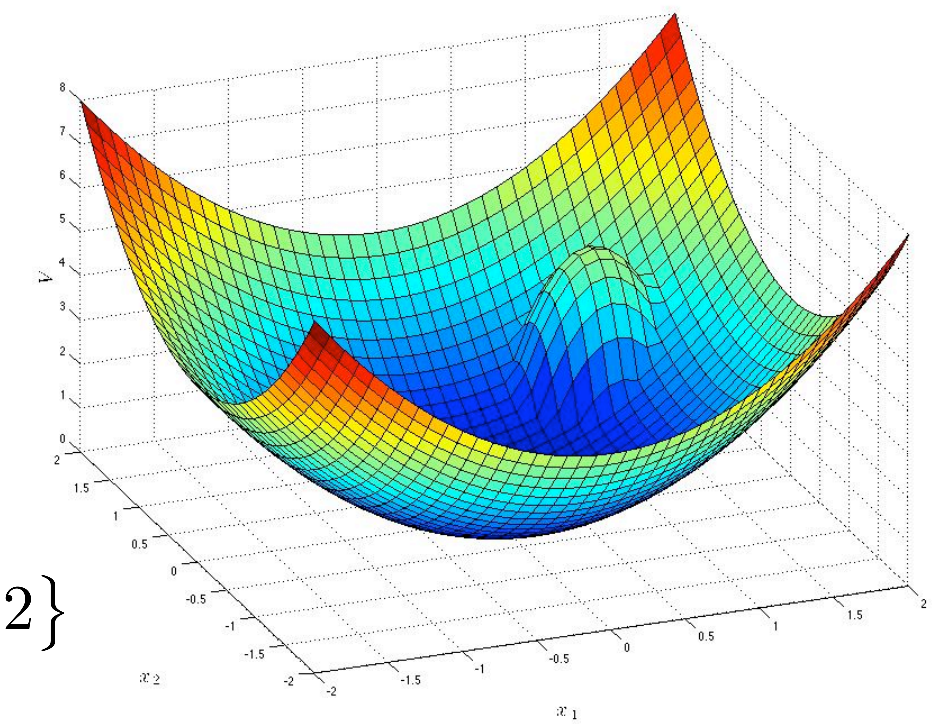

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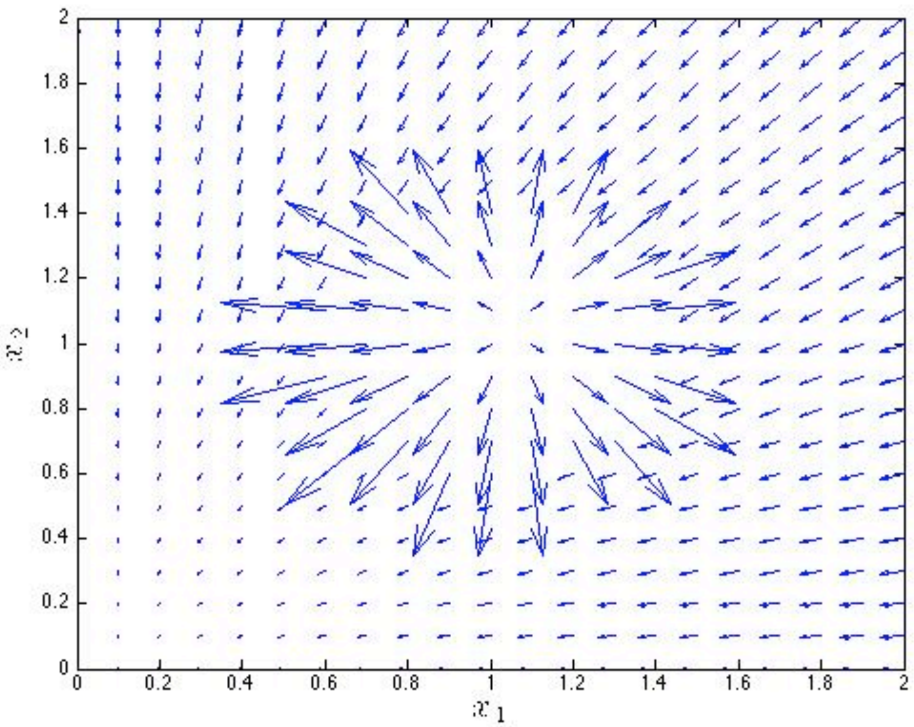
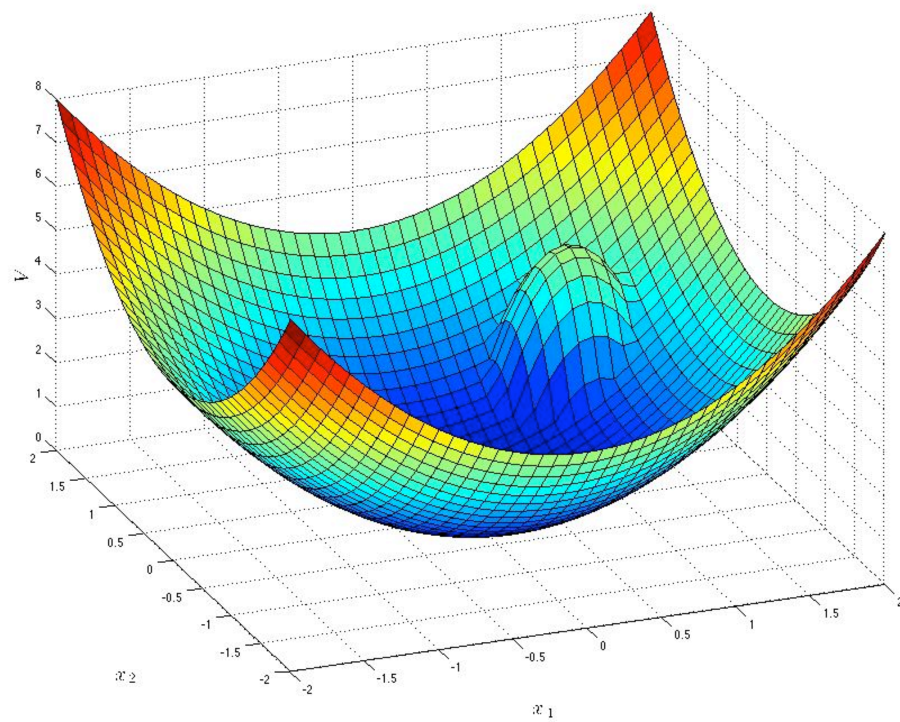
Choose $u = -L_g V(x)$.



Complete Lyapunov Functions

Definition: A complete Lyapunov function for $\dot{x} = f(x)$ is a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ which is constant on the chain-recurrent set, including attractors and repellers, and decreasing along flows elsewhere.

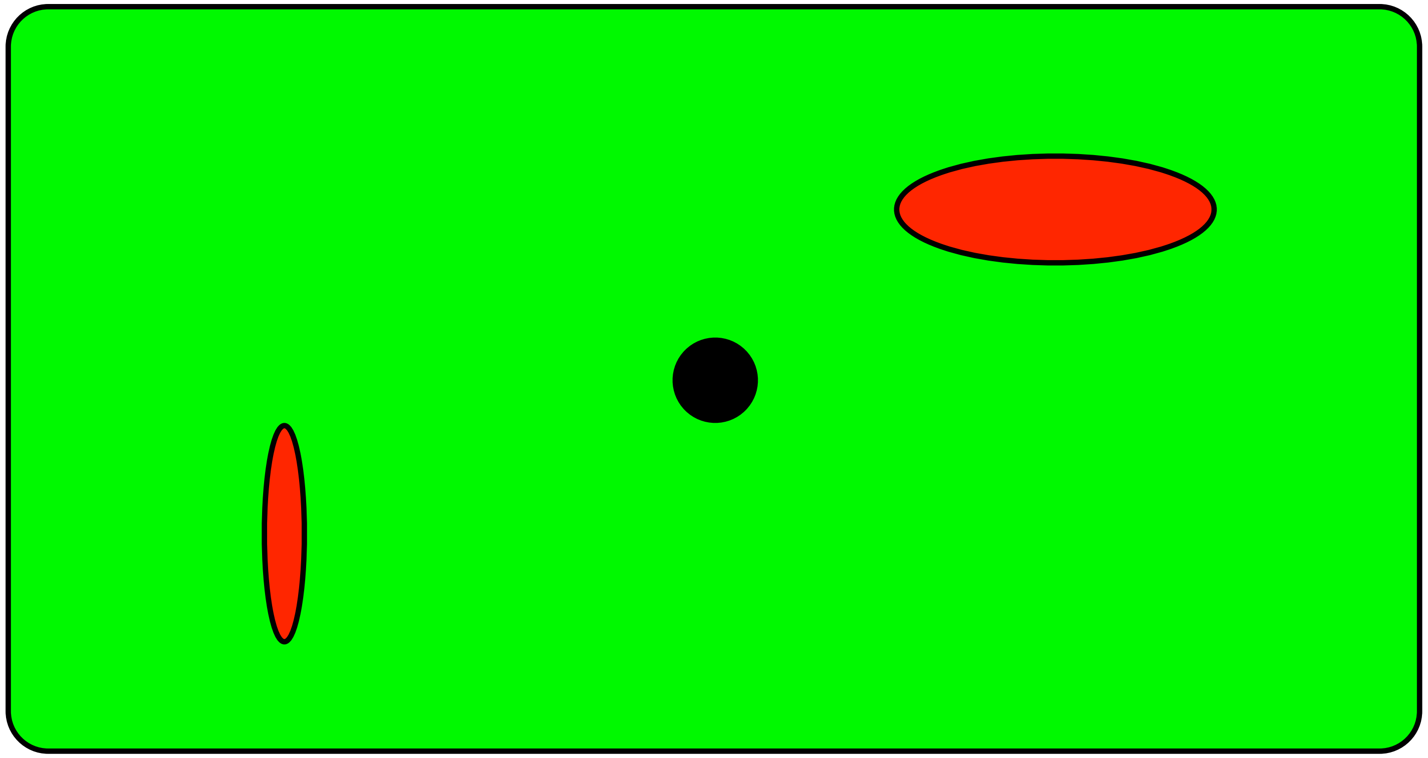
Theorem: *If Λ is a compact invariant set containing all α and ω-limit sets (plus some technical assumptions) then there exists a smooth complete Lyapunov function decreasing outside of Λ.*



Definition (and existence) of a complete control Lyapunov function?

Z. Nitecki and M. Shub, "Filtrations, Decompositions, and Explosions", *American J. Math.*, 1975.

Topological Perplexity



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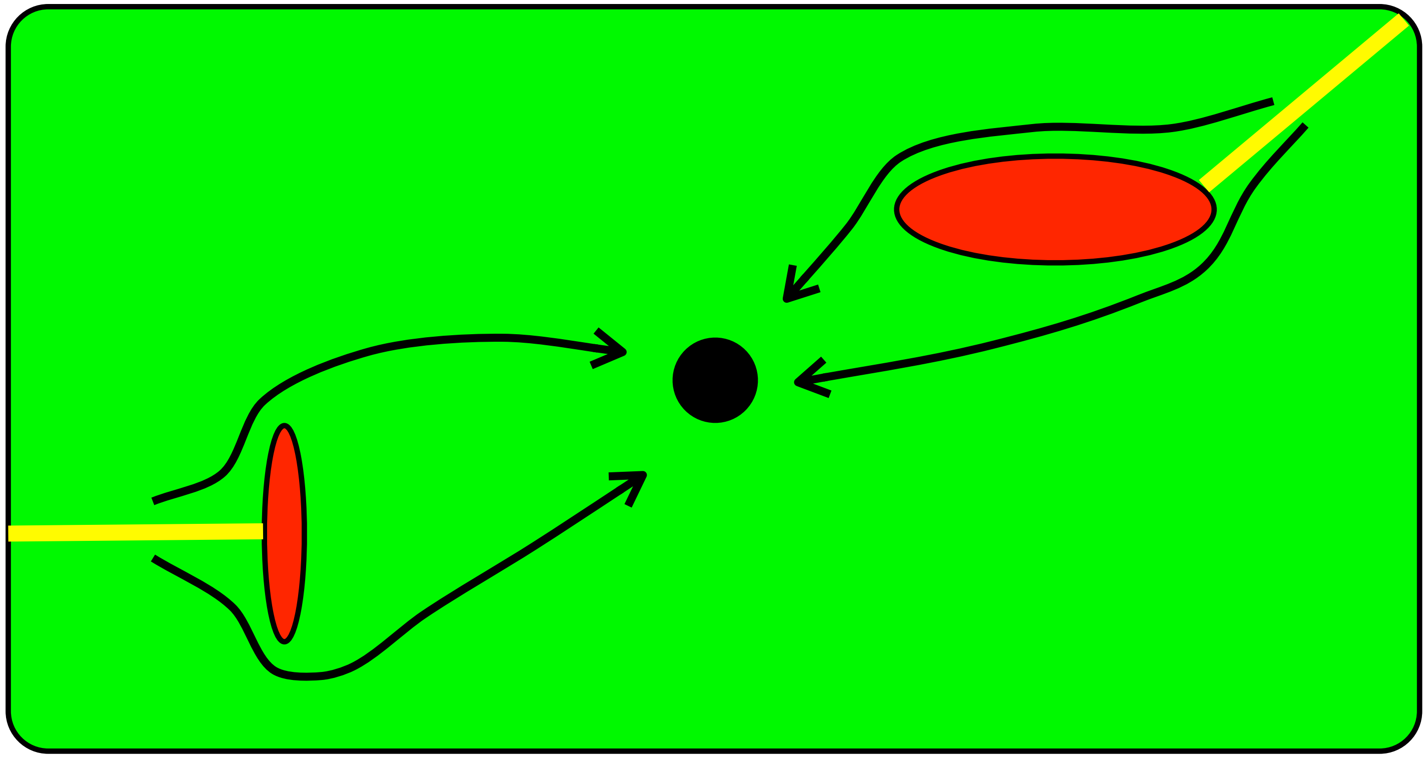

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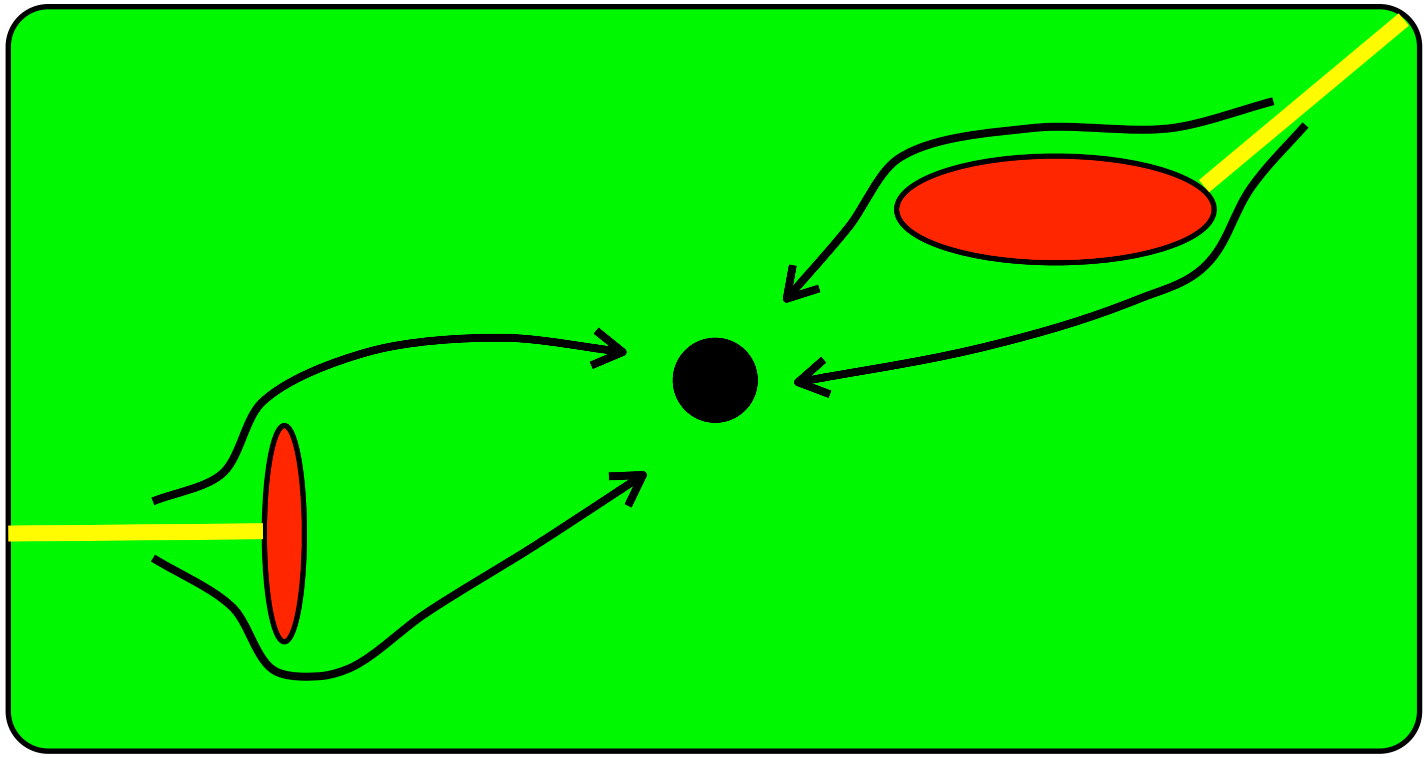

Topological Perplexity (Baryshnikov): The sum total of the Betti numbers.

A lower bound on the decision space, or, how often do I really have to choose a direction?

Summary

- Lyapunov-based feedback design
  - Necessity of nonsmooth Lyapunov functions (and discontinuous feedback)
- Destabilising constraints
  - Patching feedback controllers, Complete Lyapunov Functions, Topological Perplexity