Computing Bandpass Prolates

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Summary

Spectral concentration problem
- Bandlimited signals
- Slepian functions

Computing prolates
- Lucky accident and consequences

Computing bandpass prolates
- No lucky accident
- Prolate series
- Modulated prolate series
- Bessel series

Fourier transforms and bandlimited signals

Fourier transform: $f \in L^1(\mathbb{R})$

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi it\xi}\,dt.$$

Bandlimited signals: Given $\Omega > 0$, let

$$\mathrm{PW}_\Omega = \{f \in L^2(\mathbb{R});\ \hat{f}(\xi) = 0 \text{ for } |\xi| > \Omega/2\}.$$

Spectral concentration problem

Let $T, \Omega > 0$.

$$\lambda = \sup\{\int_{-T}^{T} |f(t)|^2 \, dt; \; f \in PW_\Omega, \; \|f\|_2 = 1\}.$$

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Time- and bandlimiting projections

$$Q_T f(t) = \mathbf{1}_{[-T,T]}(t) f(t), \quad P_\Omega f(t) = (\mathbf{1}_{[-\Omega/2,\Omega/2]} \hat{f})\check{}(t).$$

$P_\Omega Q_T : \mathrm{PW}_\Omega \to \mathrm{PW}_\Omega$ is self-adjoint, compact.

Eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_n \cdots > 0$

Eigenfunctions $\phi_0, \phi_1, \ldots, \phi_n, \ldots$ (Slepian functions)

Slepian properties

$\{\phi_j\}_{j=0}^{\infty}$ an o.n.b. for for PW_Ω

Double orthogonality: $\{\overline{\phi}_j = \lambda_j^{-1/2} Q_T \phi_j\}_{j=0}^{\infty}$ o.n.b. for $L^2[-T, T]$.

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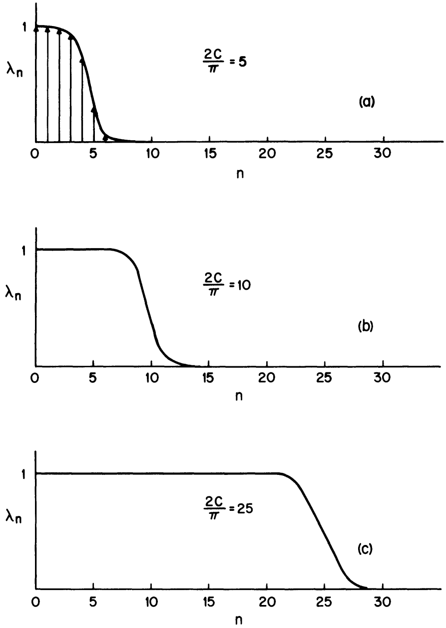
Fourier self-similarity: $\hat{\phi}_j\left(\frac{c\xi}{T}\right) = \pm i^j \sqrt{\frac{T}{c\lambda_j}} Q_T \phi_j(\xi)$

Theorem (Landau and Widom 1980)

If $c = 2\Omega T$, there are :

- *approximately c eigenvalues ≈ 1*
- *approximately $\log c$ eigenvalues in the "plunge region"*
- *after that the eigenvalues decay quickly:* $\lambda_n \sim 2\pi(c/4)^{2n+1}/(n!)^2$

Numerical difficulties in solving $P_\Omega Q_T \phi_j = \gamma_j \phi_j$.


1
λn
2C/π = 5
(a)
0 5 10 15 20 25 30
n
1
λn
2C/π = 10
(b)
0 5 10 15 20 25 30
n
1
λn
2C/π = 25
(c)
0 5 10 15 20 25 30
n

The "lucky accident"

Let

$$\mathcal{P}_c y = (1 - t^2)y'' - 2ty' + c^2 t^2 y.$$

Solutions of

$$\mathcal{P}_c \psi = \chi \psi \tag{1}$$

are known as PSWFs of order zero $\{\psi_n^{(c)}\}$ (or *prolates*).

Eigenvalues χ_n of $\mathcal{P}_c^{(m)}$ are well-separated $\rightarrow$ efficient Galerkin methods for computation of prolates:

$$\psi_j = \sum_{k=0}^{\infty} B_{jk} \overline{P}_k$$

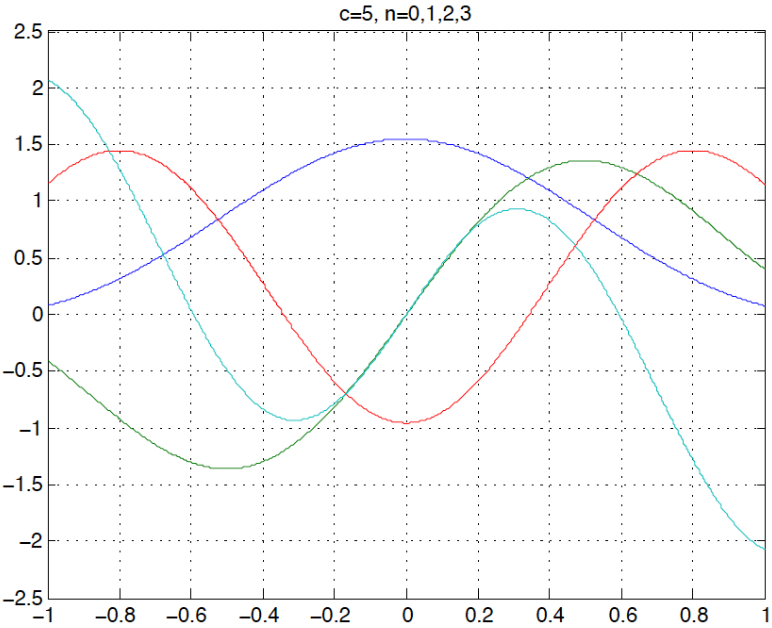
The "lucky accident"

Theorem (Slepian and Pollak, BSTJ (1) 1961)

$\mathcal{P}_c^{(0)}$ *commutes with* $P_c Q_1$

Corollary

$\psi_{0,n}^{(c)} = \phi_n^{(1,c/2)}$, *i.e., Slepians are prolates!*


c=5, n=0,1,2,3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

Bandpass prolates

Let $0 < c' < c < \infty$, $P_{c,c'} = P_{2c} - P'_{2c}$ and

$$PW_{c,c'} = \{f \in PW_c;\ \hat{f}(\xi) = 0 \text{ for } |\xi| < c'\} = PW_c \ominus PW_{c'}.$$

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A bandpass prolate is an eigenfunction of $P_{c,c'}Q_1$:

$$P_{c,c'}Q_1 f(t) = \int_{-1}^{1} \frac{sin(a(t-s))\cos(b(t-s))}{b(t-s)} f(s)\, ds$$

with $a = \dfrac{(c+c')}{2}$, $b = \dfrac{(c-c')}{2}$

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Out of luck

Theorem (Morrison: QAM (21) 1963)

There is no second or fourth order self-adjoint linear differential operator with polynomial coefficients which commutes with L

Method 1: Prolate series

We seek eigenfunctions ψ of $P_{c,c'}Q_1$.

Theorem (H., Lakey: JFAA (19) 2013)

If ψ is an eigenfunction of $P_{c,c'}Q$ with eigenvalue μ then $\psi = \sum_{n=0}^{\infty} a_n \phi_n^{(c)}$ with $\mathbf{a} = (a_0, a_1, \dots, a_n, \dots)^T$ satisfying

$$\mu \mathbf{a} = (I - R)\Lambda \mathbf{a}$$

where $\Lambda = diag(\lambda_0, \lambda_1, \dots)$ and $R_{jn} = \langle P_{c'}\phi_n^{(c)}, \phi_j^{(c)} \rangle$

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$$R_{jn} = \int_{-c'}^{c'} \widehat{\phi_n^{(c)}}(\xi)\overline{\widehat{\phi_j^{(c)}}}(\xi)\, d\xi = \frac{i^{k-j}}{\sqrt{\lambda_j \lambda_k}} \int_{-c'/c}^{c'/c} \phi_n^{(c)}(t)\phi_j^{(c)}(t)\, dt$$

Computing bandpass prolates

Byerly identities:

$$(j(j+1) - n(n+1)) \int_a^b P_n P_j = (t^2 - 1)(P_n' P_j - P_j' P_n)\Big|_{t=a}^{t=b}$$

$$(\chi_n - \chi_j) \int_a^b \phi_n \phi_j = (t^2 - 1)(\phi_n' \phi_j - \phi_j' \phi_n)\Big|_{t=a}^{t=b}$$

$$\phi_n = \sum_j C_{nj} P_j \Rightarrow \int_a^b \phi_n^2 = \sum_{j,k} C_{nj} C_{nk} \int_a^b P_j P_k$$

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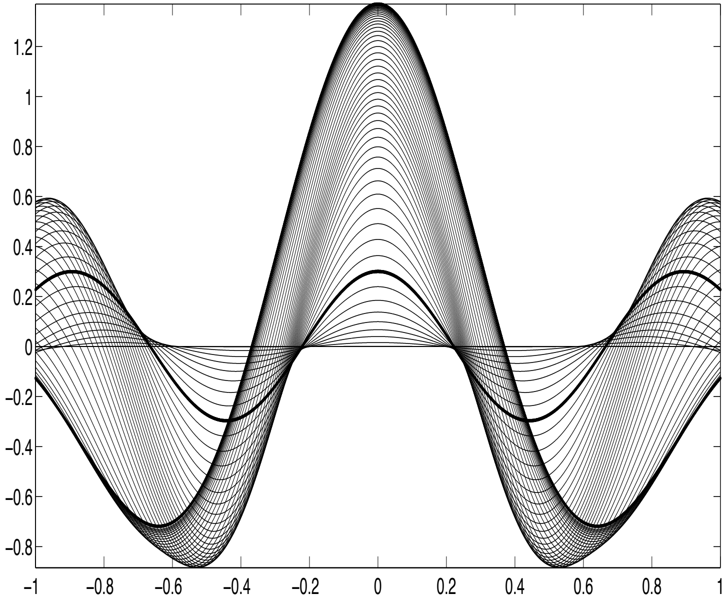
$$\int_a^b P_n^2 = \frac{(2n-1)}{(2n+1)} \int_a^b P_{n-1}^2 + \int_a^b P_n P_{n-2}$$
$$- \frac{(2n-1)}{(2n+1)} \int_a^b P_{n-1} P_{n+1} + (2n-1) Q_{n-1} Q_n \Big|_a^b$$

where $Q_n = \dfrac{1}{2n+1}(P_{n+1} - P_{n-1})$.


1.2
1
0.8
0.6
0.4
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0
−0.2
−0.4
−0.6
−0.8
−1
−0.8
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1

Method 2: Modulated prolate series

With $\sigma = \pm$, consider the functions

$$\phi_{n,\sigma}^{(c,c')}(t) = e^{2\pi i \sigma a t} \phi_n^{(b)}(t) \qquad (b = c - c').$$

The collection $\{\phi_{n,\sigma}^{(c,c')};\, n \geq 0,\ \sigma = \pm\}$ is an o.n.b. for $PW_{c,c'}$.
We seek eigenfunctions of the form $f = \sum_{n,\sigma} a_n^\sigma \phi_{n,\sigma}^{(c,c')}$:

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$$\begin{aligned}
\lambda f = P_{c,c'}Q_1 f &= \sum_{n,\sigma} a_n^\sigma P_{c,c'}Q_1\phi_{n,\sigma}^{(c,c')} \\
&= \sum_{n,\sigma} a_n^\sigma \sum_{m,\sigma'} \langle P_{c,c'}Q_1\phi_{n,\sigma}^{(c,c')}, \phi_{m,\sigma'}^{(c,c')}\rangle \phi_{m,\sigma'}^{(c,c')} \\
&= \sum_{n,\sigma} a_{n,\sigma} \sum_{m,\sigma'} \langle Q_1\phi_{n,\sigma}^{(c,c')}, \phi_{m,\sigma'}^{(c,c')}\rangle \phi_{m,\sigma'}^{(c,c')} \\
&= \sum_{n,\sigma} a_{n,\sigma} \sum_{m,\sigma'} C_{n,m}^{(\sigma,\sigma')}\phi_{m,\sigma'}^{(c,c')}.
\end{aligned}$$

Computing bandpass prolates

Equivalently

$$Ca^\sigma = \lambda a^\sigma$$

with

$$C_{n,m}^{(\sigma,\sigma')} = \int_{-1}^{1} \phi_{n,\sigma}^{(c,c')}(t)\overline{\phi_{m,\sigma'}^{(c,c')}(t)}\, dt.$$

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$$C_{n,m}^{(\sigma,\sigma)} = \int_{-1}^{1} \phi_{n}^{(b)}(t)\overline{\phi_{m}^{(b)}(t)}\,dt = \lambda_n \delta_{n,m}$$

$$C_{n,m}^{(+,-)} = \int_{-1}^{1} e^{4\pi i a t}\phi_{n}^{(b)}(t)\overline{\phi_{m}^{(b)}(t)}\,dt$$

Computing bandpass prolates

$$C_{n,m}(s) = \int_{-1}^{1} e^{2\pi i s t} \phi_n(t) \phi_m(t)\, dt.$$

Then

$$\frac{d}{ds} C(s) = 2\pi i A C(s) \qquad C(0) = I$$

where

$$A_{n,\ell} = \int_{-1}^{1} u \phi_n(u) \phi_\ell(u)\, du = \frac{\pi}{c} \sqrt{\lambda_n \lambda_\ell} \frac{(1 + (-1)^{n+\ell})}{\lambda_n + \lambda_\ell} \psi_n(1) \psi_\ell(1).$$

so

$$C(s) = e^{2\pi i s A}$$

Method 3

$$C(s) = \sqrt{\Lambda} B e^{2\pi i s Q} B^T \sqrt{\Lambda}$$

where

$$Q_{jk} = \int_{-1}^{1} t \overline{P}_j(t) \overline{P}_k(t)\, dt.$$

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Theorem

If $n \leq N$, then

$$T_N(Q^n) = T_N[(T_{2N}Q)^n]$$

$$\left\| T_N\left(\sum_{n=0}^{N} \frac{(2\pi i s (T_{2N}Q))^n}{n!} \right) - T_N(e^{2\pi i s Q}) \right\|_{\ell^2 \to \ell^2} \leq 2 \frac{(2\pi s)^N}{N!}$$

Method 4: Bessel series

$$P_n(z)P_m(z) = \sum_{k=0}^{m} \frac{a_{m-k}a_k a_{n-k}}{a_{n+m-k}} \left(\frac{2n+2m-4k+1}{2n+2m-2k+1} \right) P_{n+m-2k}(z)$$

where $a_k = \dfrac{(2k-1)!!}{k!} = \dfrac{(2k)!}{2^k (k!)^2}$.

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Theorem

With Q as above and $C_{n,m,k} = \frac{a_{m-k}a_k a_{n-k}}{a_{n+m-k}} \left(\frac{2n+2m-4k+1}{2n+2m-2k+1} \right)$

$$(e^{isQ})_{mn} = \sum_{k=0}^{m} \sqrt{(n+\frac{1}{2})(m+\frac{1}{2})} C_{n,m,k} \int_{-1}^{1} e^{ist} P_{n+m-2k}(t)\, dt$$

$$= \sqrt{2\pi(n+\frac{1}{2})(m+\frac{1}{2})} i^{n+m} \sum_{k=0}^{m} (-1)^k C_{n,m,k} s^{1/2} J_{1/2+n+m-2k}(s)$$

Enjoy your sabbatical!