On class number relations and intersections over function fields

Number Theory Online Conference (5 June 2020)

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(Joint work with Jia-Wei Guo, working in progress)

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Classical story

Kronecker-Hurwitz class number relation

Given a negative integer D with $D \equiv 0$ or $1 \bmod 4$, let

$$O_D := \mathbb{Z}\left[\frac{D+\sqrt{D}}{2}\right], \quad w(D) := \frac{\#(O_D^\times)}{2}, \quad h(D) := \text{class number of } O_D.$$

The Hurwitz class number is:

$$H(D) := \sum_{\substack{c \in \mathbb{N} \\ c^2 \mid D}} \frac{h(D/c^2)}{w(D/c^2)}.$$

Put $H(0) := -1/12 \; (= \zeta_{\mathbb{Q}}(-1))$.

Kronecker-Hurwitz class number relation

Theorem (Kronecker (1860), Gierster (1880), Hurwitz (1885))

For $n \in \mathbb{N}$, we have

$$\sum_{\substack{t \in \mathbb{Z} \\ t^2 \leq 4n}} H(t^2 - 4n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \max(d, n/d).$$

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Let $\mathfrak{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$. Let $X := \mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}^*$. Given $m \in \mathbb{N}$, let

$$\mathcal{Z}_m := \mathrm{Im}\Big(X_0(m) := \Gamma_0(m) \backslash \mathfrak{H}^* \longrightarrow X \times X\Big)$$

under the map $\big([z] \mapsto ([z], [mz])\big)$. For $n \in \mathbb{N}$, consider the divisor

$$\mathcal{Z}(n) := \sum_{\substack{d \in \mathbb{N} \\ d^2 \mid n}} \mathcal{Z}_{n/d^2}.$$

Geometric interpretation

Put $\mathcal{Z} = \mathcal{Z}(1)$. Then for a non-square $n \in \mathbb{N}$,

$$\mathcal{Z} \cdot \mathcal{Z}(n) = \Big(X \cdot \mathcal{Z}(n)\Big)_f + \Big(X \cdot \mathcal{Z}(n)\Big)_\infty,$$

where

$$\sum_{\substack{t \in \mathbb{Z} \\ t^2 \leq 4n}} H(t^2 - 4n) = \Big(\mathcal{Z} \cdot \mathcal{Z}(n)\Big)_f = \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \max(d, n/d),$$

and

$$\Big(\mathcal{Z} \cdot \mathcal{Z}(n)\Big)_\infty = \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \min(d, n/d).$$

In particular,

$$\mathcal{Z} \cdot \mathcal{Z}(n) = 2\sigma(n), \quad \text{where } \sigma(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d.$$

Connection with Eisenstein series

Recall the following weight 2 Eisenstein series:

$$E_2(z) \quad = \quad -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z}, \quad z \in \mathfrak{H},$$

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$$\begin{aligned} E_2(z) &= -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z}, \quad z \in \mathfrak{H}, \\ &= \frac{1}{2}\left(-\frac{\mathrm{vol}(\mathcal{Z})}{2} + \sum_{n=1}^{\infty} \big(\mathcal{Z} \cdot \mathcal{Z}(n)\big) e^{2\pi i n z}\right). \end{aligned}$$

(Here $\mathrm{vol}(\mathcal{Z}) = 1/6 = -2\zeta_{\mathbb{Q}}(-1)$.)

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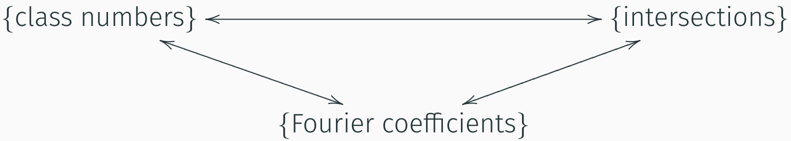

Hirzebruch-Zagier class number relation

Let p be a prime number with $p \equiv 1 \bmod 4$, $F := \mathbb{Q}(\sqrt{p})$, and O_F the ring of integers in F. Consider the Hilbert modular surface $\mathcal{S}_F := \mathrm{SL}_2(O_F)\backslash(\mathfrak{H} \times \mathfrak{H})$, Hirzebruch and Zagier introduce a family of special curves $\{\mathcal{Z}(n) \mid n \in \mathbb{N}\}$ on $\mathcal{S}_F$, and show the connection between their intersections and class numbers: for $n \in \mathbb{N}$, put

$$G_p(n) := \sum_{\substack{t \in \mathbb{Z} \\ t^2 \leq 4n \text{ and } p \mid t^2 - 4n}} H\left(\frac{t^2 - 4n}{p}\right) \quad \text{and } I_p(n) := \frac{1}{\sqrt{p}} \sum_{\substack{\lambda \in O_F^+ \\ \lambda\lambda' = n}} \min(\lambda, \lambda').$$

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Theorem (Hirzebruch-Zagier)

$$\mathcal{Z}(1) \cdot \mathcal{Z}(n) = G_p(n) + I_p(n), \quad \forall n \in \mathbb{N},$$

and

$$\varphi_p(z) := -\frac{\mathrm{vol}(\mathcal{Z}(1))}{2} + \sum_{n=1}^{\infty} \big(\mathcal{Z}(1) \cdot \mathcal{Z}(n)\big) e^{2\pi i n z}, \quad z \in \mathfrak{H}$$

is a weight-2 modular form of Nebentypus $\left(\frac{\cdot}{p}\right)$ for $\Gamma_0(p)$.

Function field setting

Notations

- $A := \mathbb{F}_q[\theta]$ (with q odd)
- $A_+ := \{\text{monic } f \in A\}$
- $k := \mathbb{F}_q(\theta)$
- $|a/b| := q^{\deg a - \deg b}$ for $a, b \in A$ with $b \neq 0$
- $k_\infty := \mathbb{F}_q(\!(\theta^{-1})\!)$
- $O_\infty := \mathbb{F}_q[\![\theta^{-1}]\!]$
- $\mathbb{C}_\infty := \widehat{\overline{k}}_\infty$
- $\pi_\infty := \theta^{-1} \in O_\infty$

$$(A_+, A, k, k_\infty, \mathbb{C}_\infty) \longleftrightarrow (\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C})$$

Let $D \in A$ with $D \prec 0$ (i.e. the place ∞ of k does not split in $k(\sqrt{D})$). Let $O_D := A[\sqrt{D}]$, $w(D) := \#(O_D^\times)/(q-1)$, and $h(D)$ be the class number of O_D. Let

$$H(D) := \sum_{\substack{\mathfrak{c} \in A_+ \\ \mathfrak{c}^2 \mid D}} \frac{h(D/\mathfrak{c}^2)}{w(D/\mathfrak{c}^2)} \quad \text{and} \quad H(0) = -\frac{1}{q^2-1} \; (= \zeta_A(-1)).$$

Theorem (Wang-Yu, J.-K. Yu)

Given a non-square $\mathfrak{n} \in A_+$, we have

$$\sum_{\epsilon \in \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2} \sum_{\substack{t \in A \\ t^2 \preceq 4\epsilon\mathfrak{n}}} H(t^2 - 4\epsilon\mathfrak{n}) = \sum_{\substack{\mathfrak{d} \in A_+ \\ \mathfrak{d} \mid \mathfrak{n}}} \max\left(|\mathfrak{d}|, |\mathfrak{n}/\mathfrak{d}|\right) - |\mathfrak{n}|^{1/2} \sum_{\substack{\mathfrak{d} \in A_+ \\ \mathfrak{d} \mid \mathfrak{n},\ 2\deg \mathfrak{d} = \deg \mathfrak{n}}} \frac{|\mathfrak{n}| - |\mathfrak{n} - \mathfrak{d}^2|}{q-1}.$$

Connection with intersections

Theorem (J.-K. Yu)

Given a non-square $\mathfrak{n} \in A_+$,

$$\sum_{\epsilon \in \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2} \sum_{\substack{t \in A \\ t^2 \preceq 4\epsilon\mathfrak{n}}} H(t^2 - 4\epsilon\mathfrak{n}) = \left(\mathcal{Z} \cdot \mathcal{Z}(\mathfrak{n})\right)_f,$$

Here $\mathcal{Z}$ is the diagonal image of X into $X \times X$, and $\mathcal{Z}(\mathfrak{n})$ is the graph coming from the Hecke correspondence on the Drinfeld modular curve X of full level. Moreover, $\mathcal{Z} \cdot \mathcal{Z}(\mathfrak{n}) = 2\sigma(\mathfrak{n})$, where $\sigma(\mathfrak{n}) := \sum_{\mathfrak{d}|\mathfrak{n}} |\mathfrak{d}|$.

Connection with Fourier coefficients

Remark: We have Gekeler's "improper" Eisenstein series:

$$E(x,y) = |y| \sum_{\substack{a\in A \\ \deg a + 2 \leq \mathrm{ord}_\infty(y)}} \sigma(a)\psi(ax), \quad (x,y) \in k_\infty \times k_\infty^\times,$$

where $\sigma(0) := 1/(1-q^2)$, and $\psi : k_\infty \to \mathbb{C}^\times$ is defined by

$$\psi(\sum_i \epsilon_i \pi_\infty^i) := \exp\left(\frac{2\pi\sqrt{-1}}{p}\mathrm{Trace}_{\mathbb{F}_q/\mathbb{F}_p}(-\epsilon_1)\right).$$

Put $\mathrm{vol}(\mathcal{Z}) := 2/(q^2-1)$. Then:

$$E(x,y) = \frac{|y|}{2} \cdot \left[-\mathrm{vol}(\mathcal{Z}) + \sum_{\substack{a\in A-\{0\} \\ \deg a \leq \mathrm{ord}_\infty(y)}} (\mathcal{Z}\cdot\mathcal{Z}(a))\psi(ax)\right].$$

Main result

Modified Hurwitz class number

Given square-free $\mathfrak{n}^+, \mathfrak{n}^- \in A_+$ with $\gcd(\mathfrak{n}^+, \mathfrak{n}^-) = 1$, for $D \in A$ with $D \prec 0$ define

$$h^{\mathfrak{n}^+,\mathfrak{n}^-}(D) := h(D) \prod_{\mathfrak{p} | \mathfrak{n}^+} \left(1 + \left\{\frac{D}{\mathfrak{p}}\right\}\right) \prod_{\mathfrak{p} | \mathfrak{n}^-} \left(1 - \left\{\frac{D}{\mathfrak{p}}\right\}\right).$$

Here

$$\left\{\frac{D}{\mathfrak{p}}\right\} := \begin{cases} -1 & \text{if } \mathfrak{p} \nmid D \text{ and } \mathfrak{p} \text{ is inert in } k(\sqrt{D}), \\ 0 & \text{if } \mathfrak{p} \| D, \\ 1 & \text{otherwise.} \end{cases}$$

Class number relation

Let

$$H^{\mathfrak{n}^+,\mathfrak{n}^-}(D) := \sum_{\substack{\mathfrak{c}\in A_+\\ \mathfrak{c}^2\mid D}} \frac{h^{\mathfrak{n}^+,\mathfrak{n}^-}(D/\mathfrak{c}^2)}{w(D/\mathfrak{c}^2)} \quad \text{and} \quad H^{\mathfrak{n}^+,\mathfrak{n}^-}(0) := \frac{\prod_{\mathfrak{p}\mid\mathfrak{n}^\pm}(|\mathfrak{p}|\pm 1)}{1-q^2}.$$

Theorem I (Guo-W.)

Let $\mathfrak{p}_0 \in A_+$, irreducible with $\deg \mathfrak{p}_0$ even. Given square-free $\mathfrak{n}^+, \mathfrak{n}^- \in A_+$ such that $\left(\frac{\mathfrak{p}_0}{\mathfrak{q}}\right) = \pm 1$ for every prime $\mathfrak{q} \mid \mathfrak{n}^\pm$, suppose $\mathfrak{n}^-$ has even number of prime factors. Then for non-zero $a \in A$, we have

$$2\sum_{\substack{t\in A\\ t^2\preceq 4a}} H^{\mathfrak{p}_0\mathfrak{n}^+,\mathfrak{n}^-}(\mathfrak{p}_0(t^2-4a)) = \mathcal{Z}\cdot\mathcal{Z}(a),$$

where $\mathcal{Z}$ and $\mathcal{Z}(a)$ are “special divisors” on “Drinfeld-Stuhler modular surface”.

Drinfeld-Stuhler modular surface

Let $\Omega := \mathbb{C}_\infty - k_\infty$, the Drinfeld half plane, equipped with the Möbius action of $\mathrm{GL}_2(k_\infty)$. Let $F = k(\sqrt{\mathfrak{p}_0})$ (∞ splits in F). The embedding $F \hookrightarrow F \otimes_k k_\infty \cong k_\infty \times k_\infty$ where $\alpha \in F$ maps to (α, α'), induces an embedding $\mathrm{GL}_2(F) \hookrightarrow \mathrm{GL}_2(k_\infty)^2$. We then have an action of $\mathrm{GL}_2(F)$ on $\Omega \times \Omega =: \Omega_F$.

Let $\mathfrak{n} := \mathfrak{n}^+ \cdot \mathfrak{n}^-$, and

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_F) \,\middle|\, ad - bc \in \mathbb{F}_q^\times,\ \mathfrak{n} \mid c \right\}.$$

The Drinfeld-Stuhler modular surface for Γ is $\mathcal{S}_\Gamma := \Gamma \backslash \Omega_F$.

(moduli space of “Frobenius-Hecke sheaves”)

Base curves

Let $B := \left(\frac{\mathfrak{p}_0, \mathfrak{n}}{k}\right) = k + k\mathbf{i} + k\mathbf{j} + k\mathbf{ij}$ with $\mathbf{i}^2 = \mathfrak{p}_0$, $\mathbf{j}^2 = \mathfrak{n}$, and $\mathbf{ij} = -\mathbf{ji}$. Then B is the quaternion algebra over k ramified precisely at the primes dividing $\mathfrak{n}^-$. We have embeddings $F \hookrightarrow B \hookrightarrow \mathrm{Mat}_2(F)$ ($\cong B \otimes_k F$):

$$\sqrt{\mathfrak{p}_0} \mapsto \mathbf{i} \mapsto \begin{pmatrix} \sqrt{\mathfrak{p}_0} & 0 \\ 0 & -\sqrt{\mathfrak{p}_0} \end{pmatrix} \quad \text{and} \quad \mathbf{j} \mapsto \begin{pmatrix} 0 & 1 \\ \mathfrak{n} & 0 \end{pmatrix}.$$

Put $O_B := \mathrm{Mat}_2(O_F) \cap B$, an Eichler A-order of type $(\mathfrak{p}_0\mathfrak{n}^+, \mathfrak{n}^-)$. Let $X_{\mathfrak{n}^-}(\mathfrak{p}_0\mathfrak{n}^+) := O_B^\times \backslash \Omega$, the modular curve of "$B$-elliptic sheaves" (introduced by Laumon-Rapoport-Stuhler).

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Natural map from $X_{\mathfrak{n}^-}(\mathfrak{p}_0\mathfrak{n}^+)$ to $\mathcal{S}_\Gamma$:

$$[z] \in O_B^\times \backslash \Omega \longmapsto [z, S_1 z] \in \Gamma \backslash \Omega_F, \qquad \text{where } S_1 := \begin{pmatrix} 0 & 1 \\ \mathfrak{n} & 0 \end{pmatrix}.$$

The image of $X_{\mathfrak{n}^-}(\mathfrak{p}_0\mathfrak{n}^+)$ in $\mathcal{S}_\Gamma$ is denoted by $\mathcal{Z}$.

Special divisors

Let $V := \left\{ \begin{pmatrix} a & \alpha \\ -\mathfrak{n}\alpha' & b \end{pmatrix} \,\middle|\, a, b \in k,\ \alpha \in F \right\}$, and $\Lambda := V \cap \mathrm{Mat}_2(O_F)$.
Given non-zero $a \in A$ and $x \in \Lambda$ with $\det x = a$, put

$$B_x := \left\{ b \in \mathrm{Mat}_2(F) \,\middle|\, bxb^* = \det b \cdot x \right\} \quad \text{and} \quad \Gamma_x := B_x \cap \Gamma.$$

Here $b^* := S_1^{-1}\bar{b}'S_1$ for $b \in \mathrm{Mat}_2(F)$. We have the map from $X_x := \Gamma_x \backslash \Omega$ to $\mathcal{S}_\Gamma$ defined by

$$[z] \in X_x \longmapsto [z, S_x z] \in \mathcal{S}_\Gamma, \quad \text{where } S_x := S_1\bar{x}.$$

The image of X_x in $\mathcal{S}_\Gamma$ is denoted by $\mathcal{Z}_x$. The special divisor $\mathcal{Z}(a)$ is defined to be

$$\mathcal{Z}(a) := \sum_{x \in \Gamma \backslash \Lambda_a} \mathcal{Z}_x.$$

where $\Lambda_a := \{x \in \Lambda \mid \det x = a\}$.

Let

$$P := \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \,\middle|\, (x, y) \in k_\infty \times k_\infty^\times \right\} \leq \mathrm{GL}_2(k_\infty).$$

Put

$$\mathrm{vol}(\mathcal{Z}) := \frac{2}{q^2 - 1} \cdot (|\mathfrak{p}_0| + 1) \cdot \prod_{\mathfrak{p} | \mathfrak{n}^\pm} (|\mathfrak{p}| \pm 1) = -2H^{\mathfrak{p}_0\mathfrak{n}^+, \mathfrak{n}^-}(0).$$

We have

Theorem II (Guo-W.)

The function $\mathcal{E}$ on P defined by

$$\mathcal{E}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = |y| \cdot \left[-\mathrm{vol}(\mathcal{Z}) + \sum_{\substack{a \in A - \{0\} \\ \deg a + 2 \leq \mathrm{ord}_\infty(y)}} \big(\mathcal{Z} \cdot \mathcal{Z}(a)\big) \psi(ax) \right]$$

can be extended to a "Drinfeld-type" automorphic form of nebentypus $\left(\frac{\cdot}{\mathfrak{p}_0}\right)$ for $\Gamma_0^{(1)}(\mathfrak{n}\mathfrak{p}_0)$ on $\mathrm{GL}_2(k_\infty)$.

Bridge: theta series

Theta series

Let $\mathbb{A}$ be the adele ring of k and $V_{\mathbb{A}} := V \otimes_k \mathbb{A}$. let ω_V be the Weil representation of $\mathrm{SL}_2(\mathbb{A}) \times \mathrm{O}(V)(\mathbb{A})$ on the space $S(V_{\mathbb{A}})$ of Schwartz functions on $V_{\mathbb{A}}$. For $\varphi \in S(V_{\mathbb{A}})$ and $g \in \mathrm{SL}_2(\mathbb{A})$, define

$$I(g;\varphi) := \int_{B^\times \mathbb{A}^\times \backslash B_{\mathbb{A}}^\times} \left(\sum_{x \in V} (\omega_V(g, h_b)\varphi)(x) \right) db.$$

Here $h_b \in \mathrm{O}(V)$ is defined by $h_b(x) := bxb^{-1}$ for every $x \in V$.

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Here $h_b \in \mathrm{O}(V)$ is defined by $h_b(x) := bxb^{-1}$ for every $x \in V$.
For $(x, y) \in \mathbb{A} \times \mathbb{A}^\times$, we have the Fourier expansion

$$I\left(\begin{pmatrix} y & xy^{-1} \\ 0 & y^{-1} \end{pmatrix}; \varphi \right) = \sum_{a \in k} I^*(a, y; \varphi)\psi(ax),$$

where

$$\begin{aligned} I^*(a, y; \varphi) &:= \int_{k \backslash \mathbb{A}} I\left(\begin{pmatrix} y & uy^{-1} \\ 0 & y^{-1} \end{pmatrix}; \varphi \right) \psi(-au)du \\ &= |y|_{\mathbb{A}}^2 \int_{B^\times \mathbb{A} \backslash B_{\mathbb{A}}^\times} \left(\sum_{x \in V_a} \varphi(yb^{-1}xb) \right) db. \end{aligned}$$

Connection with class numbers

Lemma 1

For $a \in k$ and $y \in \mathbb{A}^\times$, we get

$$I^*(a, y; \varphi) = |y|_{\mathbb{A}}^2 \sum_{x \in B^\times \backslash V_a} \mathrm{vol}(K_x^\times \mathbb{A}^\times \backslash K_{x,\mathbb{A}}^\times) \cdot \int_{K_{x,\mathbb{A}}^\times \backslash B_{\mathbb{A}}^\times} \varphi(yb^{-1}xb)db.$$

Here $K_x = \{b \in B \mid bx = xb\}$, and the volume is with respect to the Tamagawa measure, i.e. $\mathrm{vol}(K_x^\times \mathbb{A}^\times \backslash K_{x,\mathbb{A}}^\times) = 2L(1, \chi_{K_x})$.

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Let $\varphi_\Lambda := \varphi_\Lambda^\infty \otimes \varphi_\infty \in S(V_{\mathbb{A}})$, where

$$\varphi_\Lambda^\infty := \mathbf{1}_{\widehat{\Lambda}} \quad \text{and} \quad \varphi_\infty := \mathbf{1}_{O_{B_\infty}} - \frac{q+1}{2}\mathbf{1}_{O'_{B_\infty}}.$$

Here O_{B_∞} is a chosen maximal compact subring of $B_\infty \cong \mathrm{Mat}_2(k_\infty) \cong V_\infty$, and O'_{B_∞} is an Iwahori O_∞-order in O_{B_∞}.

Connection with class numbers

Lemma 2

For $y \in k_\infty^\times$, the Fourier coefficient $I^*(a, y; \varphi_\Lambda) = 0$ unless $a \in A$ and $\deg a + 2 \leq \mathrm{ord}_\infty(y)$. In this case,

$$I^*(a, y; \varphi_\Lambda) = \mathrm{vol}(O_{B_\mathbb{A}}^\times / O_\mathbb{A}^\times) \cdot |y|^2 \cdot \sum_{\substack{t \in A \\ t^2 \preceq 4a}} H^{\mathfrak{p}_0 \mathfrak{n}^+, \mathfrak{n}^-}(\mathfrak{p}_0(t^2 - 4a)).$$

Here the volume $\mathrm{vol}(O_{B_\mathbb{A}}^\times / O_\mathbb{A}^\times)$ (with respect to the Tamagawa measure on $B_\mathbb{A}^\times / \mathbb{A}^\times$) is euqal to

$$\frac{(q-1)(q^2-1)}{(|\mathfrak{p}_0| + 1) \prod_{\mathfrak{p} | \mathfrak{n}^\pm} (|\mathfrak{p}| \pm 1)} = \frac{1-q}{H^{\mathfrak{p}_0 \mathfrak{n}^+, \mathfrak{n}^-}(0)}.$$

On the other hand, from the strong approximation theorem we get

$$B^\times \mathbb{A}^\times \backslash B_{\mathbb{A}}^\times / \widehat{O}_B^\times \xleftarrow{\ \sim\ } O_B^\times k_\infty^\times \backslash B_\infty^\times.$$

This enables us to show

Lemma 3

For $a \in A$, $y \in k_\infty^\times$ with $\deg a + 2 \leq \operatorname{ord}_\infty(y)$,

$$I^*(a, y; \varphi_\Lambda) \ = \ |y|^2 \sum_{x \in \Gamma \backslash \Lambda_a} \mathcal{I}(x),$$

where

$$\mathcal{I}(x) := \operatorname{vol}(\widehat{O}_B^\times / \widehat{A}^\times) \sum_{\gamma \in \Gamma_x \backslash \Gamma / O_B^\times} \int_{O_B^\times \cap \gamma^{-1}\Gamma_x \gamma \backslash B_\infty^\times / k_\infty^\times} \varphi_\infty(yb^{-1}(\gamma^{-1} \star x)b)db,$$

and $\gamma^{-1} \star x := \left(\gamma^{-1} x (\gamma^*)^{-1}\right) \cdot \det(\gamma)$.

Connection with intersection numbers

Finally, the theory of local optimal embeddings assures:

$$\mathrm{vol}(\widehat{O}_B^\times/\widehat{A}^\times)\int_{O_B^\times\cap\gamma^{-1}\Gamma_x\gamma\backslash B_\infty^\times/k_\infty^\times}\varphi_\infty(yb^{-1}(\gamma^{-1}\star x)b)db$$

$$= \quad \mathrm{vol}(O_{B_{\mathbb{A}}}^\times/O_{\mathbb{A}}^\times)\cdot\begin{cases} H^{\mathfrak{p}_0\mathfrak{n}^+,\mathfrak{n}^-}(0), & \text{if } K_{\gamma^{-1}\star x}=B;\\ (q-1)/\#(O_B^\times\cap\gamma^{-1}\Gamma_x\gamma) & \text{if } K_{\gamma^{-1}\star x}/k \text{ is imaginary}\\ 0 & \text{otherwise.}\end{cases}$$

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$$\begin{aligned} & \mathrm{vol}(\widehat{O}_B^\times/\widehat{A}^\times) \int_{O_B^\times \cap \gamma^{-1}\Gamma_x\gamma \backslash B_\infty^\times / k_\infty^\times} \varphi_\infty(yb^{-1}(\gamma^{-1} \star x)b)db \\ = \quad & \mathrm{vol}(O_{B_{\mathbb{A}}}^\times/O_{\mathbb{A}}^\times) \cdot \begin{cases} H^{\mathfrak{p}_0\mathfrak{n}^+,\mathfrak{n}^-}(0), & \text{if } K_{\gamma^{-1}\star x} = B; \\ (q-1)/\#(O_B^\times \cap \gamma^{-1}\Gamma_x\gamma) & \text{if } K_{\gamma^{-1}\star x}/k \text{ is imaginary} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 4

$$\mathcal{I}(x) = \mathrm{vol}(O_{B_{\mathbb{A}}}^\times/O_{\mathbb{A}}^\times) \cdot \frac{(\mathcal{Z} \cdot \mathcal{Z}_x)}{2}.$$

Consequently,

$$I^*(a, y; \varphi_\Lambda) = |y|^2 \mathrm{vol}(O_{B_{\mathbb{A}}}^\times/O_{\mathbb{A}}^\times) \cdot \frac{(\mathcal{Z} \cdot \mathcal{Z}(a))}{2}.$$

The end. Thank you for your attention.