Discovery vs Proof, and Visual Intuition

Mathematical Thinking Workshop 2022

Aidan Sims
University of Wollongong

University of Newcastle, September 27, 2022

UNIVERSITY
OF WOLLONGONG
AUSTRALIA

Mathematical thinking (for me)

Discovery versus proof

Visual intuition - pros and cons

UNIVERSITY
OF WOLLONGONG
AUSTRALIA

Mathematical thinking: an example

Some background:

- Topological group: group with a locally compact Hausdorff topology with continuous group operation.

Mathematical thinking: an example

Some background:

- Topological group: group with a locally compact Hausdorff topology with continuous group operation.
- Fundamental group: loops based at $\star$. That is, continuous $f : [0, 1] \to X$ with $f(0) = f(1) = \star$, modulo deformation.

Mathematical thinking: an example

Some background:

- Topological group: group with a locally compact Hausdorff topology with continuous group operation.
- Fundamental group: loops based at $\star$. That is, continuous $f : [0,1] \to X$ with $f(0) = f(1) = \star$, modulo deformation.
  - Group operation is concatenation $\circ$ of loops

Mathematical thinking: an example

Some background:

- Topological group: group with a locally compact Hausdorff topology with continuous group operation.
- Fundamental group: loops based at $\star$. That is, continuous $f : [0,1] \to X$ with $f(0) = f(1) = \star$, modulo deformation.
  - Group operation is concatenation $\circ$ of loops

Fact: the fundamental group of a topological group is abelian.

Mathematical thinking: an example

Some background:

- Topological group: group with a locally compact Hausdorff topology with continuous group operation.
- Fundamental group: loops based at $\star$. That is, continuous $f : [0,1] \to X$ with $f(0) = f(1) = \star$, modulo deformation.
  - Group operation is concatenation $\circ$ of loops

Fact: the fundamental group of a topological group is abelian.

How to understand why?

UNIVERSITY OF WOLLONGONG AUSTRALIA

Mathematical thinking: an example

Some background:

- Topological group: group with a locally compact Hausdorff topology with continuous group operation.
- Fundamental group: loops based at $\star$. That is, continuous $f : [0, 1] \to X$ with $f(0) = f(1) = \star$, modulo deformation.
  - Group operation is concatenation $\circ$ of loops

Fact: the fundamental group of a topological group is abelian.

How to understand why?

Lots to think about: continuity? Use sequential arguments? Build explicit deformations from $f \circ g$ to $g \circ f$?

Eckmann–Hilton principle

My favourite explanation:

Eckmann–Hilton principle

My favourite explanation:

Two operations in play:

Eckmann–Hilton principle

My favourite explanation:

Two operations in play:

- concatenation $\circ$ of loops: $(f \circ g)(t) = \begin{cases} f(2t) & t \leq \frac{1}{2}, \\ g(2t-1) & t \geq \frac{1}{2} \end{cases}$

Eckmann–Hilton principle

My favourite explanation:

Two operations in play:

- concatenation $\circ$ of loops: $(f \circ g)(t) = \begin{cases} f(2t) & t \leq \frac{1}{2}, \\ g(2t-1) & t \geq \frac{1}{2} \end{cases}$
- pointwise multiplication $*$ of loops: $(f \cdot g)(t) = f(t)g(t)$.

Eckmann–Hilton principle

My favourite explanation:

Two operations in play:

- concatenation $\circ$ of loops: $(f \circ g)(t) = \begin{cases} f(2t) & t \leq \frac{1}{2}, \\ g(2t-1) & t \geq \frac{1}{2} \end{cases}$
- pointwise multiplication $*$ of loops: $(f \cdot g)(t) = f(t)g(t)$.

They are linked:

$$(f \circ g) * (h \circ k) = (f * h) \circ (g * k).$$

UNIVERSITY OF WOLLONGONG AUSTRALIA

Eckmann–Hilton principle

My favourite explanation:

Two operations in play:

- concatenation $\circ$ of loops: $(f \circ g)(t) = \begin{cases} f(2t) & t \le \frac{1}{2}, \\ g(2t-1) & t \ge \frac{1}{2} \end{cases}$
- pointwise multiplication $*$ of loops: $(f \cdot g)(t) = f(t)g(t)$.

They are linked:

$$(f \circ g) * (h \circ k) = (f * h) \circ (g * k).$$

Gives a quaternary operation $\begin{matrix} f & g \\ h & k \end{matrix} = \begin{matrix} (f \circ h) \\ * \\ (g \circ k) \end{matrix} = \begin{pmatrix} f \\ * \\ g \end{pmatrix} \circ \begin{pmatrix} h \\ * \\ k \end{pmatrix}$.

UNIVERSITY
OF WOLLONGONG
AUSTRALIA

Eckmann–Hilton continued

Claim: If H is a set with binary operations $\circ$ and $*$ that admits an identity 1_* for $*$ and $1_\circ$ for $\circ$ and satisfies

$$(f \circ g) * (h \circ k) = (f * h) \circ (g * k).$$

then $1_* = 1_\circ$, $* = \circ$ and it is abelian.

Eckmann–Hilton continued

Claim: If H is a set with binary operations $\circ$ and $*$ that admits an identity 1_* for $*$ and $1_\circ$ for $\circ$ and satisfies

$$(f \circ g) * (h \circ k) = (f * h) \circ (g * k).$$

then $1_* = 1_\circ$, $* = \circ$ and it is abelian.

$$\begin{matrix} 1_* & 1_\circ \\ 1_\circ & 1_* \end{matrix}$$

Eckmann–Hilton continued

Claim: If H is a set with binary operations $\circ$ and $*$ that admits an identity 1_* for $*$ and $1_\circ$ for $\circ$ and satisfies

$$(f \circ g) * (h \circ k) = (f * h) \circ (g * k).$$

then $1_* = 1_\circ$, $* = \circ$ and it is abelian.

$$1_\circ = \begin{matrix} 1_\circ \\ \circ \\ 1_\circ \end{matrix} = \begin{matrix} 1_* & 1_\circ \\ 1_\circ & 1_* \end{matrix}$$

Eckmann–Hilton continued

Claim: If H is a set with binary operations $\circ$ and $*$ that admits an identity 1_* for $*$ and $1_\circ$ for $\circ$ and satisfies

$$(f \circ g) * (h \circ k) = (f * h) \circ (g * k).$$

then $1_* = 1_\circ$, $* = \circ$ and it is abelian.

$$1_\circ = \begin{smallmatrix} 1_\circ \\ \circ \\ 1_\circ \end{smallmatrix} = \begin{matrix} 1_* & 1_\circ \\ 1_\circ & 1_* \end{matrix} = 1_* * 1_* = 1_*.$$

Eckmann–Hilton continued

Claim: If H is a set with binary operations $\circ$ and $*$ that admits an identity 1_* for $*$ and $1_\circ$ for $\circ$ and satisfies

$$(f \circ g) * (h \circ k) = (f * h) \circ (g * k).$$

then $1_* = 1_\circ$, $* = \circ$ and it is abelian.

$$1_\circ = \begin{matrix} 1_\circ \\ \circ \\ 1_\circ \end{matrix} = \begin{matrix} 1_* & 1_\circ \\ 1_\circ & 1_* \end{matrix} = 1_* * 1_* = 1_*.$$

$$\begin{matrix} a & 1 \\ 1 & b \end{matrix}$$

UNIVERSITY OF WOLLONGONG AUSTRALIA

Eckmann–Hilton continued

Claim: If H is a set with binary operations $\circ$ and $*$ that admits an identity 1_* for $*$ and $1_\circ$ for $\circ$ and satisfies

$$(f \circ g) * (h \circ k) = (f * h) \circ (g * k).$$

then $1_* = 1_\circ$, $* = \circ$ and it is abelian.

$$1_\circ = \begin{smallmatrix} 1_\circ \\ \circ \\ 1_\circ \end{smallmatrix} = \begin{matrix} 1_* & 1_\circ \\ 1_\circ & 1_* \end{matrix} = 1_* * 1_* = 1_*.$$

$$a \circ b = \begin{smallmatrix} a & 1 \\ 1 & b \end{smallmatrix}$$

Eckmann–Hilton continued

Claim: If H is a set with binary operations $\circ$ and $*$ that admits an identity 1_* for $*$ and $1_\circ$ for $\circ$ and satisfies

$$(f \circ g) * (h \circ k) = (f * h) \circ (g * k).$$

then $1_* = 1_\circ$, $* = \circ$ and it is abelian.

$$1_\circ = \begin{smallmatrix} 1_\circ \\ \circ \\ 1_\circ \end{smallmatrix} = \begin{matrix} 1_* & 1_\circ \\ 1_\circ & 1_* \end{matrix} = 1_* * 1_* = 1_*.$$

$$a \circ b = \begin{smallmatrix} a & 1 \\ 1 & b \end{smallmatrix} = a * b.$$

UNIVERSITY OF WOLLONGONG AUSTRALIA

Eckmann–Hilton continued

Claim: If H is a set with binary operations $\circ$ and $*$ that admits an identity 1_* for $*$ and $1_\circ$ for $\circ$ and satisfies

$$(f \circ g) * (h \circ k) = (f * h) \circ (g * k).$$

then $1_* = 1_\circ$, $* = \circ$ and it is abelian.

$$1_\circ = \begin{smallmatrix} 1_\circ \\ \circ \\ 1_\circ \end{smallmatrix} = \begin{matrix} 1_* & 1_\circ \\ 1_\circ & 1_* \end{matrix} = 1_* * 1_* = 1_*.$$

$$a \circ b = \begin{smallmatrix} a & 1 \\ 1 & b \end{smallmatrix} = a * b.$$

$$\begin{smallmatrix} 1 & a \\ b & 1 \end{smallmatrix}$$

UNIVERSITY OF WOLLONGONG AUSTRALIA

Eckmann–Hilton continued

Claim: If H is a set with binary operations $\circ$ and $*$ that admits an identity 1_* for $*$ and $1_\circ$ for $\circ$ and satisfies

$$(f \circ g) * (h \circ k) = (f * h) \circ (g * k).$$

then $1_* = 1_\circ$, $* = \circ$ and it is abelian.

$$1_\circ = \begin{smallmatrix} 1_\circ \\ \circ \\ 1_\circ \end{smallmatrix} = \begin{matrix} 1_* & 1_\circ \\ 1_\circ & 1_* \end{matrix} = 1_* * 1_* = 1_*.$$

$$a \circ b = \begin{smallmatrix} a & 1 \\ 1 & b \end{smallmatrix} = a * b.$$

$$a \circ b = \begin{smallmatrix} 1 & a \\ b & 1 \end{smallmatrix}$$

Eckmann–Hilton continued

Claim: If H is a set with binary operations $\circ$ and $*$ that admits an identity 1_* for $*$ and $1_\circ$ for $\circ$ and satisfies

$$(f \circ g) * (h \circ k) = (f * h) \circ (g * k).$$

then $1_* = 1_\circ$, $* = \circ$ and it is abelian.

$$1_\circ = \begin{smallmatrix} 1_\circ \\ \circ \\ 1_\circ \end{smallmatrix} = \begin{matrix} 1_* & 1_\circ \\ 1_\circ & 1_* \end{matrix} = 1_* * 1_* = 1_*.$$

$$a \circ b = \begin{smallmatrix} a & 1 \\ 1 & b \end{smallmatrix} = a * b.$$

$$a \circ b = \begin{smallmatrix} 1 & a \\ b & 1 \end{smallmatrix} = b * a.$$

What's the point

Clever argument, but what's the point?

What's the point

Clever argument, but what's the point?

Exhibits many hallmarks of mathematical thinking, and of mathematical presentation

What's the point

Clever argument, but what's the point?

Exhibits many hallmarks of mathematical thinking, and of mathematical presentation

- We identified a very clear problem.

What's the point

Clever argument, but what's the point?

Exhibits many hallmarks of mathematical thinking, and of mathematical presentation

- We identified a very clear problem.
- We created a formal/symbolic approach to the problem.

What's the point

Clever argument, but what's the point?

Exhibits many hallmarks of mathematical thinking, and of mathematical presentation

- We identified a very clear problem.
- We created a formal/symbolic approach to the problem.
- We did not include all specifics in our formalism.

What's the point

Clever argument, but what's the point?

Exhibits many hallmarks of mathematical thinking, and of mathematical presentation

- We identified a very clear problem.
- We created a formal/symbolic approach to the problem.
- We did not include all specifics in our formalism.
- We explored the limits of the formal reasoning.

Mathematical thinking

For me, mathematical thinking:

- Encodes problems in formalised/idealised language

Mathematical thinking

For me, mathematical thinking:

- Encodes problems in formalised/idealised language
- Carefully identifies the rules, assumptions of the formalism

Mathematical thinking

For me, mathematical thinking:

- Encodes problems in formalised/idealised language
- Carefully identifies the rules, assumptions of the formalism
- Uses abstraction/formalism to test intuition/analogy

Mathematical thinking

For me, mathematical thinking:

- Encodes problems in formalised/idealised language
- Carefully identifies the rules, assumptions of the formalism
- Uses abstraction/formalism to test intuition/analogy
- Uses intuition/analogy to inform formal thinking

Mathematical thinking

For me, mathematical thinking:

- Encodes problems in formalised/idealised language
- Carefully identifies the rules, assumptions of the formalism
- Uses abstraction/formalism to test intuition/analogy
- Uses intuition/analogy to inform formal thinking
- Seeks diverse formalisms to tap different intuition

Mathematical thinking

For me, mathematical thinking:

- Encodes problems in formalised/idealised language
- Carefully identifies the rules, assumptions of the formalism
- Uses abstraction/formalism to test intuition/analogy
- Uses intuition/analogy to inform formal thinking
- Seeks diverse formalisms to tap different intuition
- Tests formal conclusions against concrete examples.

Using computers I

How can we employ computers?

Using computers I

How can we employ computers?

Obvious use: as in Four-Colour Theorem:

- Create a formalism for the problem.
- Use the formalism to reduce to problem to a finite number of cases that must be checked.
- Automate the enumeration and checking of the cases.

This automates the checking, but not the mathematical thinking

Using computers II

Less obvious use: pattern recognition

- Solve small examples by hand
- Generate some numerical data
- Ask computers to recognise a pattern and generating formula
- Look for hints in the formula to inform formal solution.

Example: usage II (from work with Kumjian, Pask, Whittaker)

Higher-rank graph: directed graph, but edges have colours, and blue–red paths match up with red–blue paths to form commuting squares, cubes etc.

Example: usage II (from work with Kumjian, Pask, Whittaker)

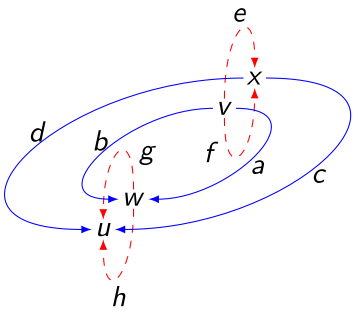
Higher-rank graph: directed graph, but edges have colours, and blue–red paths match up with red–blue paths to form commuting squares, cubes etc.

By pasting topological intervals to edges, topological squares to commuting squares, topological cubes to commuting cubes, etc, obtain topological space.

Example: usage II (from work with Kumjian, Pask, Whittaker)

Higher-rank graph: directed graph, but edges have colours, and blue–red paths match up with red–blue paths to form commuting squares, cubes etc.

By pasting topological intervals to edges, topological squares to commuting squares, topological cubes to commuting cubes, etc, obtain topological space.



UNIVERSITY OF WOLLONGONG AUSTRALIA

Example: usage II

General question: what spaces are achievable?

Example: usage II

General question: what spaces are achievable?

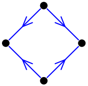
Specific question: are k-spheres (surface of $k+1$-ball) achievable?

Example: usage II

General question: what spaces are achievable?

Specific question: are k-spheres (surface of $k+1$-ball) achievable?

The 1-sphere is easy:

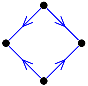


Example: usage II

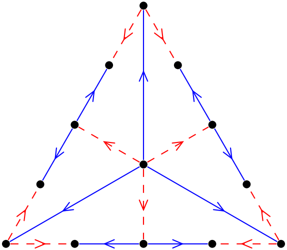
General question: what spaces are achievable?

Specific question: are k-spheres (surface of $k+1$-ball) achievable?

The 1-sphere is easy:

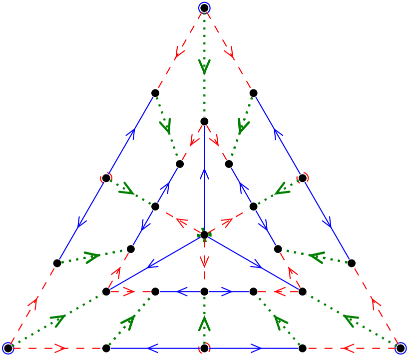


The 2-sphere slightly harder, but we could use the same idea: glue two copies of the following 2-simplex on a common boundary:



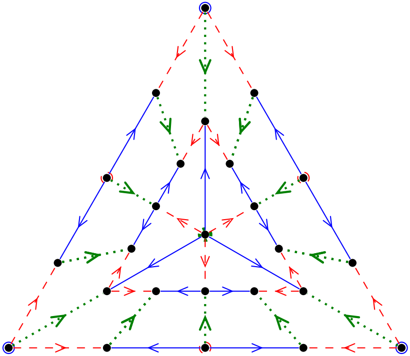
Example: usage II

The 3-sphere is getting ridicullous. I can't picture gluing two 3-spheres on a common boundary. But we found, ad hoc, a graph that worked: assemble 4 copies of the following with edges from the circled vertices (of the circle's colour) to a common central vertex.



Example: usage II

The 3-sphere is getting ridicullous. I can't picture gluing two 3-spheres on a common boundary. But we found, ad hoc, a graph that worked: assemble 4 copies of the following with edges from the circled vertices (of the circle's colour) to a common central vertex.



Now glue two of these simplices on their common boundary...

Example: usage II

4-sphere? Stuck. Something maybe inductive, but next example too big do draw, and in 4d—no visual intuition.

Example: usage II

4-sphere? Stuck. Something maybe inductive, but next example too big do draw, and in 4d—no visual intuition.

So we counted vertices in simplices: 1-simplex: 3 vertices;

Example: usage II

4-sphere? Stuck. Something maybe inductive, but next example too big do draw, and in 4d—no visual intuition.

So we counted vertices in simplices: 1-simplex: 3 vertices; 2-simplex: 13 vertices;

Example: usage II

4-sphere? Stuck. Something maybe inductive, but next example too big do draw, and in 4d—no visual intuition.

So we counted vertices in simplices: 1-simplex: 3 vertices; 2-simplex: 13 vertices; 3-simplex: 75 vertices.

Example: usage II

4-sphere? Stuck. Something maybe inductive, but next example too big do draw, and in 4d—no visual intuition.

So we counted vertices in simplices: 1-simplex: 3 vertices; 2-simplex: 13 vertices; 3-simplex: 75 vertices.

We asked a computer (specifically, the Online Encyclopedia of Integer Sequences `https://oeis.org`) about these numbers.

Example: usage II

4-sphere? Stuck. Something maybe inductive, but next example too big do draw, and in 4d—no visual intuition.

So we counted vertices in simplices: 1-simplex: 3 vertices; 2-simplex: 13 vertices; 3-simplex: 75 vertices.

We asked a computer (specifically, the Online Encyclopedia of Integer Sequences `https://oeis.org`) about these numbers.

It knew them: the number of possible outcomes of a k-horse horserace, allowing for ties. OR, the number of functions $f : \{1, \ldots, k\} \to \{1, \ldots, k\}$ such that $f(j) = \big|\{i : f(i) < f(j) \text{ for all } j\}\big|$.

Example: usage II

4-sphere? Stuck. Something maybe inductive, but next example too big do draw, and in 4d—no visual intuition.

So we counted vertices in simplices: 1-simplex: 3 vertices; 2-simplex: 13 vertices; 3-simplex: 75 vertices.

We asked a computer (specifically, the Online Encyclopedia of Integer Sequences `https://oeis.org`) about these numbers.

It knew them: the number of possible outcomes of a k-horse horserace, allowing for ties. OR, the number of functions $f : \{1, \ldots, k\} \to \{1, \ldots, k\}$ such that $f(j) = \big|\{i : f(i) < f(j) \text{ for all } j\}\big|$.

Now we could reverse-engineer labellings of vertices so that edges made sense, and solve the problem for all k.

UNIVERSITY OF WOLLONGONG AUSTRALIA

What's the point?

This wasn't a very sophisticated use of a computer.

What's the point?

This wasn't a very sophisticated use of a computer.

But it illustrates how computation can enhance mathematical thinking.

What's the point?

This wasn't a very sophisticated use of a computer.

But it illustrates how computation can enhance mathematical thinking.

Our brains seem better at finding connections between things that can be counted (*what's the relationship between the number of outcomes of a* $k+1$*-horse race and of a* k*-horse race?*) than between numbers ($3, 13, 75$; *what comes next?*)

What's the point?

This wasn't a very sophisticated use of a computer.

But it illustrates how computation can enhance mathematical thinking.

Our brains seem better at finding connections between things that can be counted (*what's the relationship between the number of outcomes of a* $k+1$*-horse race and of a* k*-horse race?*) than between numbers ($3, 13, 75$*; what comes next?*)

Computers can help us with the latter.

Usage III: finding formulas

A third use can be in identifying general formulas.

Usage III: finding formulas

A third use can be in identifying general formulas.

Straight to an example (from work with Mundey):

Usage III: finding formulas

A third use can be in identifying general formulas.

Straight to an example (from work with Mundey):

Given groups G, H that act on each other we can blend them in a Zappa-Szep product (like a semidirect product) $G \bowtie H$.

Usage III: finding formulas

A third use can be in identifying general formulas.

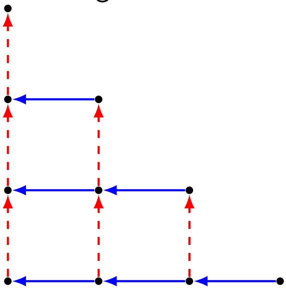
Straight to an example (from work with Mundey):

Given groups G, H that act on each other we can blend them in a Zappa-Szep product (like a semidirect product) $G \bowtie H$.

The n^{th} homology groups of $G \bowtie H$ should relate to those of G, H.

Usage III: finding formulas

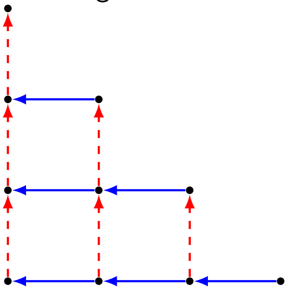
Theory says we just need to find maps between integer-valued functions on length-n staircase-shaped paths and on length-n up-across shaped paths in diagrams like:



satisfying some relations.

Usage III: finding formulas

Theory says we just need to find maps between integer-valued functions on length-n staircase-shaped paths and on length-n up-across shaped paths in diagrams like:

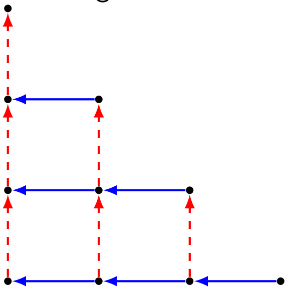


satisfying some relations.

Finding them felt like trial-and-error...

Usage III: finding formulas

Theory says we just need to find maps between integer-valued functions on length-n staircase-shaped paths and on length-n up-across shaped paths in diagrams like:



satisfying some relations.

Finding them felt like trial-and-error...

...exclusively error once n got to 3.

Usage III

But the problem is an integer linear-algebra problem with coefficients of ± 1.

Usage III

But the problem is an integer linear-algebra problem with coefficients of ± 1.

So Alex write some Python code to try all possible combinations.

Usage III

But the problem is an integer linear-algebra problem with coefficients of ± 1.

So Alex write some Python code to try all possible combinations.

This gave us formulas for $n = 3, 4, 5$.

Usage III

But the problem is an integer linear-algebra problem with coefficients of ± 1.

So Alex write some Python code to try all possible combinations.

This gave us formulas for $n = 3, 4, 5$.

Now with enough data points for some pattern recognition, we could guess a formula, and then verify it formally by induction.

Usage III

But the problem is an integer linear-algebra problem with coefficients of ± 1.

So Alex write some Python code to try all possible combinations.

This gave us formulas for $n = 3, 4, 5$.

Now with enough data points for some pattern recognition, we could guess a formula, and then verify it formally by induction.

How much of this could have been done by well-trained machine learning? Maybe a lot.

Visual intuition - strengths

I think visually.

Visual intuition - strengths

I think visually.

Generally speaking "Let X be a *blah*," $\longrightarrow$ mental picture.

Visual intuition - strengths

I think visually.

Generally speaking "Let X be a *blah*," $\longrightarrow$ mental picture.

If not, I have to build one.

Visual intuition - strengths

I think visually.

Generally speaking "Let X be a *blah*," $\longrightarrow$ mental picture.

If not, I have to build one.

This often helps me:

Visual intuition - strengths

I think visually.

Generally speaking "Let X be a *blah*," $\longrightarrow$ mental picture.

If not, I have to build one.

This often helps me:

- formulate hypotheses

Visual intuition - strengths

I think visually.

Generally speaking "Let X be a *blah*," $\longrightarrow$ mental picture.

If not, I have to build one.

This often helps me:

- formulate hypotheses
- construct proof strategies

Visual intuition - strengths

I think visually.

Generally speaking "Let X be a *blah*," $\longrightarrow$ mental picture.

If not, I have to build one.

This often helps me:

- formulate hypotheses
- construct proof strategies
- spot gaps in arguments.

Visual intuition - limitations

But the pictures are pretty limited.

Visual intuition - limitations

But the pictures are pretty limited.

Eg: I reckon < 10 mental pictures of topological spaces.

Visual intuition - limitations

But the pictures are pretty limited.

Eg: I reckon < 10 mental pictures of topological spaces.

I really don't understand why this small bank of pictures works.

Visual intuition: cons

I suspect my visual thinking hinders me in

Visual intuition: cons

I suspect my visual thinking hinders me in

- Quickly and accurately assimilating new ideas,

Visual intuition: cons

I suspect my visual thinking hinders me in

- Quickly and accurately assimilating new ideas,
- Recognising how broad (or limited) a concept is

Visual intuition: cons

I suspect my visual thinking hinders me in

- Quickly and accurately assimilating new ideas,
- Recognising how broad (or limited) a concept is
- Disproving "reasonable" conjectures.

Visual intuition: cons

I suspect my visual thinking hinders me in

- Quickly and accurately assimilating new ideas,
- Recognising how broad (or limited) a concept is
- Disproving "reasonable" conjectures.

Example: question "does every *minimal, effective* non-Hausdorff groupoid have property X?"

Visual intuition: cons

I suspect my visual thinking hinders me in

- Quickly and accurately assimilating new ideas,
- Recognising how broad (or limited) a concept is
- Disproving "reasonable" conjectures.

Example: question "does every *minimal, effective* non-Hausdorff groupoid have property X?"

I pictured things that failed X. Tried to make them minimal and effective.

Visual intuition: cons

I suspect my visual thinking hinders me in

- Quickly and accurately assimilating new ideas,
- Recognising how broad (or limited) a concept is
- Disproving "reasonable" conjectures.

Example: question "does every *minimal, effective* non-Hausdorff groupoid have property X?"

I pictured things that failed X. Tried to make them minimal and effective.

Couldn't;

Visual intuition: cons

I suspect my visual thinking hinders me in

- Quickly and accurately assimilating new ideas,
- Recognising how broad (or limited) a concept is
- Disproving "reasonable" conjectures.

Example: question "does every *minimal, effective* non-Hausdorff groupoid have property X?"

I pictured things that failed X. Tried to make them minimal and effective.

Couldn't; and now I could *only* "see" examples like them.

Visual intuition: cons

A clever approach (Nekrashevych):

Visual intuition: cons

A clever approach (Nekrashevych):

- describe algebraic ways of constructing complex groupoids from finite data

Visual intuition: cons

A clever approach (Nekrashevych):

- describe algebraic ways of constructing complex groupoids from finite data
- identify algebraic conditions equivalent to

Visual intuition: cons

A clever approach (Nekrashevych):

- describe algebraic ways of constructing complex groupoids from finite data
- identify algebraic conditions equivalent to
    1. Hausdorffness

Visual intuition: cons

A clever approach (Nekrashevych):

- describe algebraic ways of constructing complex groupoids from finite data
- identify algebraic conditions equivalent to
    1. Hausdorffness
    2. minimality

Visual intuition: cons

A clever approach (Nekrashevych):

- describe algebraic ways of constructing complex groupoids from finite data
- identify algebraic conditions equivalent to
  1. Hausdorffness
  2. minimality
  3. effectiveness.

Visual intuition: cons

A clever approach (Nekrashevych):

- describe algebraic ways of constructing complex groupoids from finite data
- identify algebraic conditions equivalent to
    1. Hausdorffness
    2. minimality
    3. effectiveness.
- now seek examples that satisfy (2) and (3) but not (1).

Visual intuition: cons

A clever approach (Nekrashevych):

- describe algebraic ways of constructing complex groupoids from finite data
- identify algebraic conditions equivalent to
  1. Hausdorffness
  2. minimality
  3. effectiveness.
- now seek examples that satisfy (2) and (3) but not (1). He found one!

Thoughts we cannot think?

Recently saw a quote:

Are there thoughts we cannot think?

Thoughts we cannot think?

Recently saw a quote:

Are there thoughts we cannot think?

I thought about my visual-thinking problem, Nekrashevych's solution.

Thoughts we cannot think?

Recently saw a quote:

Are there thoughts we cannot think?

I thought about my visual-thinking problem, Nekrashevych's solution.

There are definitely "pictures I cannot see."

Thoughts we cannot think?

Recently saw a quote:

Are there thoughts we cannot think?

I thought about my visual-thinking problem, Nekrashevych's solution.

There are definitely "pictures I cannot see."

Mathematical formalism/thinking can circumvent limitations on "what we can think."

Thoughts we cannot think?

Recently saw a quote:

Are there thoughts we cannot think?

I thought about my visual-thinking problem, Nekrashevych's solution.

There are definitely "pictures I cannot see."

Mathematical formalism/thinking can circumvent limitations on "what we can think."

But we usually convert to problems we can think about directly.

Thoughts we cannot think?

Recently saw a quote:

Are there thoughts we cannot think?

I thought about my visual-thinking problem, Nekrashevych's solution.

There are definitely "pictures I cannot see."

Mathematical formalism/thinking can circumvent limitations on "what we can think."

But we usually convert to problems we can think about directly.

Should we get better at finding out what computers can learn and at converting to problems they are good at?