Identities

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Jon Borwein Commemorative Conference

Number Theory, Special Functions and π

Newcastle, 25–29 September, 2017

With Gratitude

Srinivasa Ramanujan

Alexandru Zaharescu

Sun Kim

Atul Dixit

Richard Askey

Jon Borwein

The greatest of all sins is ingratitude. Ignatius, March, 1542

Ramanujan’s Passport Photo

Alexandru Zaharescu

Sun Kim

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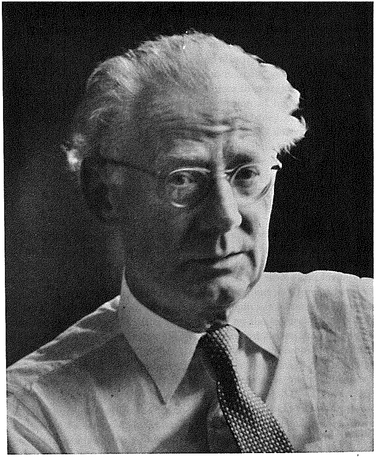
Hans Rademacher Quote

. . . the impression may have prevailed that analytic number theory deals foremost with asymptotic expressions for arithmetical functions. This view, however, overlooks another side of analytic number theory, which I may indicate by the words "identities," . . . "structural considerations." This line of research is not yet so widely known; it may very well be that methods of this type will lead to the "deeper" results, will reveal the sources of some of the results of the first direction of approach.

Hans Rademacher, September 5, 1941, address to the American Mathematical Society

Hans Rademacher Picture



Hans Rademacher

Partitions

Definition The partition function $p(n)$ is defined to be the number of ways a positive integer n can be written as a sum of positive integers.

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$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Rademacher's Primary Example

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left\{\frac{hK}{k}\left(n-\frac{1}{24}\right)^{1/2}\right\}}{\left(n-\frac{1}{24}\right)^{1/2}}\right),$$

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where

$$A_k(n) = \sum_{\substack{h \,(\mathrm{mod}\, k)\\(h,k)=1}} \omega_{h,k} e^{-2\pi hn/k}, \quad K = \pi\sqrt{2/3},$$

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$$p(1729) = 173305455943737246971728329004427554248274 0$$

q-continued fractions

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \cdots}}}}$$

$$= b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} + \cdots$$

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Rogers–Ramanujan continued fraction

$$R(q) := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots, \qquad |q| < 1$$

Leonard James Rogers

Rademacher's Second Example

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6},$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots, \quad |q|<1.$$

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$$T(q^5) - q - \frac{q^2}{T(q^5)} = \frac{(q;q)_\infty}{(q^{25};q^{25})_\infty}.$$

Circle Problem

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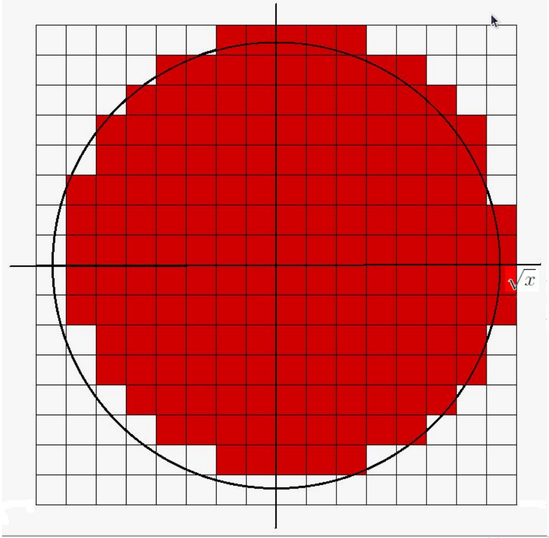
Each representation of n as a sum of two squares can be associated with a lattice point in the plane. For example, $5 = (-2)^2 + 1^2$ can be associated with the lattice point $(-2, 1)$. Then each lattice point can be associated with a unit square, say that unit square for which the lattice point is in the southwest corner.

Circle Problem



Circle Problem

$$R(x) := \sum_{0 \leq n \leq x}{}' r_2(n) = \pi x + P(x), \qquad (1)$$

where the prime $'$ on the summation sign on the left side indicates that if x is an integer, only $\frac{1}{2} r_2(x)$ is counted.

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$$R(x) < \pi(\sqrt{x} + \sqrt{2})^2,$$
$$R(x) > \pi(\sqrt{x} - \sqrt{2})^2,$$

$$R(x) = \pi x + O(\sqrt{x}) \qquad \text{Gauss}$$

G. H. Hardy

Circle Problem

Ramanujan (1914) and Hardy (1915) proved that

$$\sum_{0\leq n\leq x}{}' r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n) \left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}). \qquad (2)$$

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n}, \qquad 0 < |z| < \infty, \qquad \nu \in \mathbb{C}.$$

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$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z-\frac{1}{2}\pi\nu-\frac{1}{4}\pi\right) + O\left(\frac{1}{z^{3/2}}\right), \qquad z\to\infty.$$

Sierpinski's Theorem

$$P(x) = O(x^{1/3}), \qquad x \to \infty \qquad \text{Sierpinski (1906)}$$

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E. Landau, *Vorlesungen über Zahlentheorie, Zweiter Band*

Identity of Jacobi

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$$\begin{aligned}
\sum_{0<n\le x}{}' r_2(n) &= 4 \sum_{0<n\le x}{}' \sum_{d|n} \sin\left(\frac{\pi d}{2}\right) \\
&= 4 \sum_{0<dj\le x}{}' \sin\left(\frac{\pi d}{2}\right) \\
&= 4 \sum_{0<d\le x}{}' \left[\frac{x}{d}\right] \sin\left(\frac{\pi d}{2}\right),
\end{aligned}$$

where $[x]$ is the greatest integer $\le x$.

Ramanujan's First Claim

To state Ramanujan's claims, we need to first define

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases} \tag{3}$$

where, as customary, $[x]$ is the greatest integer less than or equal to x.

The First Claim – Now Proved by BCB, S. Kim, A. Zaharescu

Theorem

Let $F(x)$ be defined by (3), let $J_1(x)$ denote the ordinary Bessel function of order 1, let $0 < \theta < 1$, and let $x > 0$. Then

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin(2\pi n\theta) = \pi x \left(\frac{1}{2} - \theta\right) - \frac{1}{4}\cot(\pi\theta)$$
$$+\frac{1}{2}\sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\}.$$

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BCB, S. Kim and A. Zaharescu, *The circle and divisor problems, and double series of Bessel functions*, Adv. Math. **236** (2013), 24–59.

Version of BCB & Zaharescu

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BCB and A. Zaharescu, *Weighted divisor sums and Bessel function series*, Math. Ann. **335** (2006), 249–283.

Another Beautiful Identity of Ramanujan as Recorded by Hardy

If $a, b > 0$, then

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+a}} e^{-2\pi\sqrt{(n+a)b}} = \sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+b}} e^{-2\pi\sqrt{(n+b)a}} \qquad (4)$$

not given anywhere in Ramanujan's work.

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If we differentiate (4) with respect to b, let $a \to 0$, replace $2\pi\sqrt{b}$ by s, and use analytic continuation, we find that, for Re $s > 0$,

$$\sum_{n=1}^{\infty} r_2(n) e^{-s\sqrt{n}} = \frac{2\pi}{s^2} - 1 + 2\pi s \sum_{n=1}^{\infty} \frac{r_2(n)}{(s^2 + 4\pi^2 n)^{3/2}},$$

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which was the key identity in Hardy's proof of

$$P(x) = \Omega_{\pm}(x^{1/4}), \qquad \text{as } x \to \infty.$$

Explanation of Notation

There exist a sequence $\{x_n\} \to \infty$ such that

$$P(x_n) > C_1 x_n^{1/4}, \quad n \geq 1.$$

There exist a sequence $\{x'_n\} \to \infty$ such that

$$P(x'_n) < -C_2 (x'_n)^{1/4}, \quad n \geq 1.$$

Logarithmic Mean Identites

$$\sum_{n \leq x} r_2(n) \log \frac{x}{n} = \pi x - \log x + \zeta_2'(0) - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r_2(n)}{n} J_0(2\pi\sqrt{nx}),$$

$$\zeta_k(s) := \sum_{n=1}^{\infty} r_k(n) n^{-s}, \quad \sigma = \text{Re } s > \tfrac{1}{2}k.$$

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B.C. Berndt, S. Kim and A. Zaharescu, *The Circle Problem of Gauss and the Divisor Problem of Dirichlet–Still Unsolved*, Amer. Math. Monthly, to appear.

The Riemann Zeta-Function

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$$\theta(-1/z) = \sqrt{z/i}\,\theta(z)$$

Back to Sums of Squares

Let $r_k(n)$ denote the number of ways of representing the positive integer n as a sum of k squares.

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Theta Transformation Formula

$$\sum_{n=0}^{\infty} r_k(n) e^{-\pi n y} = y^{-k/2} \sum_{n=0}^{\infty} r_k(n) e^{-\pi n/y}, \qquad \text{Re } y > 0.$$

Bessel Series Identity

If $x > 0$ and $q > \frac{1}{2}(k-3)$, then

$$\frac{1}{\Gamma(q+1)} {\sum_{0 \le n \le x}}' r_k(n)(x-n)^q$$

$$= \frac{\pi^{k/2} x^{k/2+q}}{\Gamma(q+1+k/2)} + \left(\frac{1}{\pi}\right)^q \sum_{n=1}^{\infty} r_k(n) \left(\frac{x}{n}\right)^{k/4+q/2} J_{k/2+q}(2\pi\sqrt{nx}).$$

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Theorem

The functional equation, theta transformation formula, and Bessel series identity are equivalent.

An Identity of Popov

$$\frac{\pi^{k/2-1}z^{k/4-1/2}}{\Gamma(\frac{1}{2}k)} + \sum_{n=1}^{\infty} r_k(n)\frac{J_{k/2-1}(2\pi\sqrt{nz})}{n^{k/4-1/2}}e^{-\pi nt}$$

$$= \frac{e^{-\pi z/t}}{t}\left\{\frac{\pi^{k/2-1}z^{k/4-1/2}}{t^{k/2-1}\Gamma(\frac{1}{2}k)} + \sum_{n=1}^{\infty} r_k(n)\frac{I_{k/2-1}\left(\dfrac{2\pi\sqrt{nz}}{t}\right)}{n^{k/4-1/2}}e^{-\pi n/t}\right\}.$$

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$$I_\nu(z) := e^{-\pi i\nu/2}J_\nu(iz) = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{\nu+2n}, \quad 0 < |z| < \infty.$$

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$$I_\nu(z):=e^{-\pi i\nu/2}J_\nu(iz)=\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{\nu+2n},\quad 0<|z|<\infty.$$

A. Popov, *On Some Summation Formulas* (in Russian), Bull. Acad. Sci. USSR (1934), 801–802.

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Bruce Berndt, Atul Dixit, Sun Kim, and Alexandru Zaharescu, Proc. Amer. Math. Soc. **145** (2017), 3795–3808.

A. I. Popov

Alexander Ivanovich Popov (1899–1973)
He was born in the Pskov region in northwest Russia in 1899.
He graduated from Leningrad University and taught at Leningrad Polytechnic Institute.
During the period 1930–1945 he published 13 papers in mathematics.
He turned to Finno-Ugric Linguistics (group of languages in northeast Europe including Finnish, Estonian, and Hungarian languages) and wrote a doctoral thesis on toponymics.
He still taught mathematics and was the head of the Department of Mathematical Logic at the Machine-building Institute at the Leningrad State University.

A. I. Popov

A Formula of Ramanujan and Guinand

Entry (p. 253)

Let $\sigma_k(n) = \sum_{d|n} d^k$, *and let* $\zeta(s)$ *denote the Riemann zeta function. Let*

$$K_\nu(z) := \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\pi\nu)}.$$

If α *and* β *are positive numbers such that* $\alpha\beta = \pi^2$, *and if* s *is any complex number, then*

$$\sqrt{\alpha} \sum_{n=1}^{\infty} \sigma_{-s}(n) n^{s/2} K_{s/2}(2n\alpha) - \sqrt{\beta} \sum_{n=1}^{\infty} \sigma_{-s}(n) n^{s/2} K_{s/2}(2n\beta)$$
$$= \frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2} - \alpha^{(1-s)/2}\}$$
$$+ \frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\{\beta^{(1+s)/2} - \alpha^{(1+s)/2}\}. \quad (5)$$

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The identity (5) is equivalent to a formula established by Guinand

A. P. Guinand)

A Transformation Formula, Notation

$$\xi(s) := (s-1)\pi^{-\frac{1}{2}s}\Gamma(1+\tfrac{1}{2}s)\zeta(s).$$

Then Riemann's Ξ-function is defined by

$$\Xi(t) := \xi(\tfrac{1}{2}+it).$$

A Transformation Formula, Notation

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Then Riemann's Ξ-function is defined by

$$\Xi(t) := \xi(\tfrac{1}{2}+it).$$

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$$

A Transformation Formula

Entry

Define

$$\phi(x) := \psi(x) + \frac{1}{2x} - \log x. \tag{6}$$

If α *and* β *are positive numbers such that* $\alpha\beta = 1$, *then*

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty}\phi(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\phi(n\beta)\right\}$$

$$= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt, \tag{7}$$

where γ *denotes Euler's constant and* $\Xi(x)$ *denotes Riemann's* Ξ*-function.*

Remarks on this Transformation Formula

Ramanujan writes that it "can be deduced from"

Entry

If $n > 0$,

$$\int_0^\infty (\psi(1+x) - \log x) \cos(2\pi nx) dx = \frac{1}{2} (\psi(1+n) - \log n) . \quad (8)$$

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The first equality in (7) established by Guinand in 1947.

"This formula also seems to have been overlooked."

Remarks on this Transformation Formula

1. "Professor T. A. Brown tells me that he proved the self-reciprocal property of $\psi(1+x) - \log x$ some years ago, and that he communicated the result to the late Professor G. H. Hardy. Professor Hardy said that the result was also given in a progress report to the University of Madras by S. Ramanujan, but was not published elsewhere."

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2. For $|\arg z| < \pi$, as $z \to \infty$,

$$\psi(z) \sim \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots .$$

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3. Two proofs by BCB and Atul Dixit.
4. Dixit has found two further proofs, generalizations, and analogues.

The Four Tenth Order Mock Theta Functions

$$\phi_{10}(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}, \qquad \psi_{10}(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}}, \tag{9}$$

$$X_{10}(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}}, \qquad \chi_{10}(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}. \tag{10}$$

First Transformation Formula for Tenth Order Mock Theta Functions

Entry (p. 9)

If $\phi_{10}(q)$ *and* $\psi_{10}(q)$ *are defined by* (9), *then, for* $n > 0$,

$$\int_0^\infty \frac{e^{-\pi n x^2}}{\cosh \frac{2\pi x}{\sqrt{5}} + \frac{1+\sqrt{5}}{4}}\,dx + \frac{1}{\sqrt{n}} e^{\pi/(5n)} \psi_{10}(-e^{-\pi/n})$$

$$= \sqrt{\frac{5+\sqrt{5}}{2}} e^{-\pi n/5} \phi_{10}(-e^{-\pi n}) - \frac{\sqrt{5}+1}{2\sqrt{n}} e^{-\pi/(5n)} \phi_{10}(-e^{-\pi/n}). \tag{11}$$

Second Transformation Formula for Tenth Order Mock Theta Functions

Entry (p. 9)

If $\phi_{10}(q)$ *and* $\psi_{10}(q)$ *are defined by* (9), *then, for* $n > 0$,

$$\int_0^\infty \frac{e^{-\pi n x^2}}{\cosh \frac{2\pi x}{\sqrt{5}} + \frac{1-\sqrt{5}}{4}} dx + \frac{1}{\sqrt{n}} e^{\pi/(5n)} \psi_{10}(-e^{-\pi/n})$$

$$= -\sqrt{\frac{5-\sqrt{5}}{2}} e^{\pi n/5} \psi_{10}(-e^{-\pi n}) + \frac{\sqrt{5}-1}{2\sqrt{n}} e^{-\pi/(5n)} \phi_{10}(-e^{-\pi/n}). \tag{12}$$

Second Transformation Formula for Tenth Order Mock Theta Functions

Entry (p. 9)

If $\phi_{10}(q)$ *and* $\psi_{10}(q)$ *are defined by* (9), *then, for* $n > 0$,

$$\int_0^\infty \frac{e^{-\pi n x^2}}{\cosh \frac{2\pi x}{\sqrt{5}} + \frac{1-\sqrt{5}}{4}} dx + \frac{1}{\sqrt{n}} e^{\pi/(5n)} \psi_{10}(-e^{-\pi/n})$$

$$= -\sqrt{\frac{5-\sqrt{5}}{2}} e^{\pi n/5} \psi_{10}(-e^{-\pi n}) + \frac{\sqrt{5}-1}{2\sqrt{n}} e^{-\pi/(5n)} \phi_{10}(-e^{-\pi/n}). \quad (12)$$

Y.-S. Choi, *Tenth order mock theta functions in Ramanujan's lost notebook*. IV, Trans. Amer. Math. Soc. **354** (2002), 705–733.

Good-by and Thank You

Thank you for your attendance. May you prove many beautiful identities during your mathematical careers.