A short walk can be beautiful

(Based on https://www.carma.newcastle.edu.au/jon/beauty.pdf).

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ABSTRACT. The story I tell is of research undertaken, with students and colleagues, in the last six or so years on short random walks. As the research progressed, my criteria for 'beauty' changed.

Things seemingly remarkable became 'simple' and other seemingly simple things became more remarkable as our analytic and computational tools were refined, and understanding improved.

1 Introduction

Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.

– Bertrand Russell (1872-1970)[a]

[a] Quoted from *A History of Western Philosophy*, 1945

Beauty in mathematics is frequently discussed and rarely captured precisely.

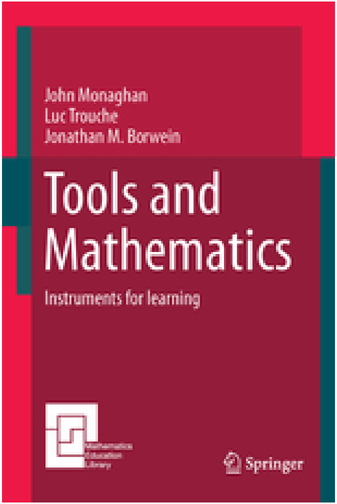
- Terms like ‘economy’, ‘elegance’, and ‘unexpectedness’, abound but mostly a researcher will say “I know it when I see it”
  - US Justice Potter Stewart on pornography (1964) [BD08]
- As Russell writes mathematics is the most austere and least accessible of the arts.
  - No one alive understands more than a small fraction of the ever growing corpus.

A century ago von Neumann is supposed to have claimed familiarity with a quarter of the subject.

- A peek at Tim Gowers' *Companion to Pure Mathematics* will show how impossible that now is.
  - Clearly — except pictorially — one can only find beautiful what one can in some sense apprehend.
- I am a pretty broadly trained and experienced researcher, but large swathes of, say, modern algebraic geometry or non commutative topology are too far from my ken for me to ever find them beautiful.

- Aesthetics also change and old questions become both unfashionable and seemingly arid (useless and/or ugly)
  - often because progress becomes too difficult as Felix Klein wrote a century ago about elliptic functions [BB87].
- Modern packages like *Maple* and *Mathematica* or the open source SAGE have made it possible to go further.
- This is both exciting and unexpected in that we tend to view century-old well studied topics as largely barren.
  - But the new *tools* are game changers.[1]

[1] This is a topic we follow up on in [MTB15].



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2 Background on walks

> *When the facts change, I change my mind. What do you do, sir?*
>
> – John Maynard Keynes (1883-1946)[a]
>
> [a] Quoted in "Keynes, the Man", The *Economist*, December 18 1996, page 47.

- An n-step uniform random walk in $\mathbb{R}^d$ starts at the origin and makes n independent steps of length 1, each taken in a uniformly random direction.
  - Thence, each step corresponds to a random vector uniformly distributed on the unit sphere.

The study of such walks originates with Pearson [Pea05].

- Pearson's interest was in planar walks, which he looked at [Pea06] as migrations of, for instance, mosquitos moving a step after each breeding cycle.
- Random walks in three dimensions ('random flights') seem first to have been studied in extenso by Rayleigh [Ray19], and higher dimensions were mentioned in [Wat41, §13.48].
- Self-avoiding random walks are now much in vogue as they model polymers and much else.

While for both random walks and their self-avoiding cousins, it is often the case that we should like to allow variable step lengths, it is only for two or three steps that we can give a closed form to the general density [Wat41].

- Thence, as often in mathematics we simplify, and in simplifying hope that we also abstract, refine, and enhance.

While we tend to think of classical areas as somehow fully understood, the truth is that we often move on because, as Klein said, progress becomes too difficult.

- Not necessarily because there is nothing important left to say.
  - New tools like new theorems can change the playing field and it is important that we teach such flexibility as suggested by Keynes.
  - Teaching students to *read* mathematical formulas is crucial.

Learning to read formulas

I now describe a small part of [BNSW11, BSW13, BSWZ12], which studied analytic and number theoretic behaviour of *short* uniform planar random walks (6 steps or less). In [BSV15] we revisited the issues in higher dimensions.

- To our surprise (pleasure), in [BSV15] we could provide complete extensions for most of the central results in the culminating paper [BSWZ12].
- The underlying mathematics is taken nearly verbatim from [BSV15] so that the interested reader finds it easy to pursue the subject in moderate detail.

Throughout n and d will denote the number of steps and dimension of the walk we are considering. We denote by ν the half-integer

$$\nu = \frac{d}{2} - 1. \tag{1}$$

Most results are more naturally expressed in terms ν, and so we denote, for instance, by

$$p_n(\nu; x)$$

the probability density function of the distance to the origin after n random unit steps in $\mathbb{R}^d$.

- We first develop basic Bessel integral representations for these densities beginning with Theorem 1.
- In Section 3 we look at the density, and in Section 4, we turn to the associated moment function

$$W_n(\nu; s) := \int_0^\infty x^s p_n(\nu; x)\mathrm{d}x. \tag{2}$$

- In particular, we derive in Theorem 5 a formula for the even moments $W_n(\nu; 2k)$
  - as a multiple sum over products of multinomial coefficients.

- This gives yet another interpretation of the ubiquitous Catalan numbers as the even moments of the distance after two random steps in four dimensions
  - and realize, more generally, in Example 8, the moments in four dimensions in terms of powers of the Narayana triangular matrix.
- We shall see that dimensions two and four are privileged in that all even moments are integral only in those two dimensions.
- In Section 5 we give an illustration of finding beauty in less trammelled places (five steps).

We recall that the general *hypergeometric function*— is given by

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| x\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{x^n}{n!} $$

and its analytic continuations.

- Here

$$ (c)_n := c(c+1)\cdots(c+n-1) $$

is the *rising factorial.*

The hypergeometric functions and the Bessel functions defined below are two of the most significant *special functions of mathematical physics* [DLMF].

- They are 'special' in that they are not *elementary* – and arise as solutions of second order algebraic differential equations.
- They are ubiquitous in our mathematical description of the physical universe.
  - Like precious stones each has its own best features and occasional flaws.

Old questions are solved by disappearing, evaporating, while new questions corresponding to the changed attitude of endeavor and preference take their place. Doubtless the greatest dissolvent in contemporary thought of old questions, the greatest precipitant of new methods, new intentions, new problems, is the one effected by the scientific revolution that found its climax in the "Origin of Species." – John Dewey (1859-1952)[a]

[a] In his 1910 essay *The influence of Darwin on Philosophy*.

3 The probability density

- [Hug95, Chapter 2.2] shows how to write the probability density $p_n(\nu; x)$ of an n-step random walk in d dimensions.

Below, the *normalized Bessel function of the first kind* is

$$j_\nu(x) = \nu! \left(\frac{2}{x}\right)^\nu J_\nu(x) = \nu! \sum_{m \geqslant 0} \frac{(-x^2/4)^m}{m!(m+\nu)!}. \qquad (3)$$

- With this normalization, we have $j_\nu(0) = 1$ and

$$j_\nu(x) \sim \frac{\nu!}{\sqrt{\pi}} \left(\frac{2}{x}\right)^{\nu+1/2} \cos\left(x - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right) \qquad (4)$$

as $x \to \infty$ on the real line.

The Bessel function is a natural generalization of exp:

$$j_0(2\sqrt{x}) = \sum_{n\geqslant 0} \frac{x^n}{n!^2} \quad \text{while} \quad \exp(x) = \sum_{n\geqslant 0} \frac{x^n}{n!}.$$

- Note also that $j_{1/2}(x) = \mathrm{sinc}(x) = \sin(x)/x$, which in part explains why analysis in 3-space is so simple.

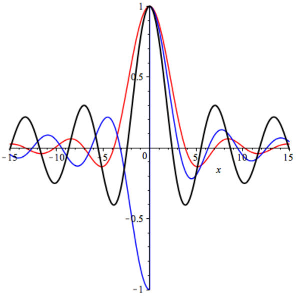


Figure 1: $j(\nu; x)$ for $\nu = 0, \frac{1}{2}, 1$

– All half-integer order $j_\nu(x)$ are elementary and so the odd dimensional theory is much simpler.

- While only 2 and 3 dimensions arise easily in physically meaningful settings, we discovered that four and higher dimensional information is needed to explain two dimensional behaviour.
  - This elegant discovery is reminiscent of how one needs complex numbers to understand real polynomials.

Theorem 1. (Bessel integrals for the densities) *The probability density function of the distance to the origin in $d \geqslant 2 (\nu \geqslant 0)$ dimensions after $n \geqslant 2$ steps is, for $x > 0$,*

$$p_n(\nu; x) = \frac{2^{-\nu}}{\nu!} \int_0^\infty (tx)^{\nu+1} J_\nu(tx) j_\nu^n(t) \mathrm{d}t. \tag{5}$$

More generally, for integer $k \geqslant 0$, and $x > 0$,

$$p_n(\nu; x) = \frac{2^{-\nu}}{\nu!} \frac{1}{x^{2k}} \int_0^\infty (tx)^{\nu+k+1} J_{\nu+k}(tx) \left(-\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\right)^k j_\nu^n(t) \mathrm{d}t. \tag{6}$$

We shall see that (6) is more tractable. It is also computationally useful.

- The densities $p_3(\nu; x)$ in dimensions $2, 3, \ldots, 9$ are shown to the left in Figure 2. In the plane, there is a logarithmic singularity at $x = 1$, otherwise the functions are at least continuous in the interval of support $[0, 3]$.
- The densities of four-step walks are shown on the right of Figure 2, and corresponding plots for five steps are shown in Figure 4.

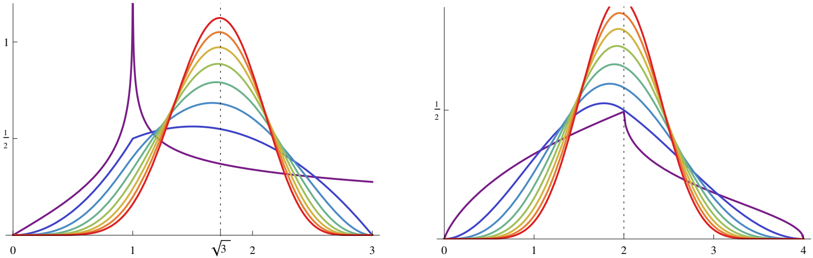


Figure 2: $p_3(\nu; x)$ and $p_4(\nu; x)$ for $\nu = 0, \frac{1}{2}, 1, \ldots, \frac{7}{2}$

The densities center and spike as d increases: $p_n(\nu; x)$ approaches a Dirac distribution centered at $\sqrt{n}$. The intuition is that as d increases, the directions tend to be close to orthogonal. Pythagoras' theorem now applies.

- These simple pictures are only easy to draw given a good computer implementation of the Bessel function
  - and reasonable plotting software!
- The striking 4-step planar density has been named "the shark-fin curve" by the late Richard Crandall.
  - This naming itself is a subversive aesthetic act — after attaching the name one can never look at the graph in the same way again
  - as is discussed in John Berger's seminal 1972 TV series and subsequent book "Ways of Seeing" http://en.wikipedia.org/wiki/Ways_of_Seeing.

Integrating (5), yields a Bessel integral representation for the cumulative distribution functions,

$$P_n(\nu; x) = \int_0^x p_n(\nu; y)\mathrm{d}y, \tag{7}$$

of the distance to the origin after n steps in d dimensions.

Corollary 2. (Cumulative distribution) *Suppose $d \geqslant 2$ and $n \geqslant 2$. Then, for $x > 0$,*

$$P_n(\nu; x) = \frac{2^{-\nu}}{\nu!} \int_0^\infty (tx)^{\nu+1} J_{\nu+1}(tx) j_\nu^n(t) \frac{\mathrm{d}t}{t}. \tag{8}$$

- So we see the Bessel function is not just computationally necessary but it is also theoretically unavoidable.

- Just as the *Airy* function is needed to understand rainbows [DLMF], these functions are often the preferred way to capture the natural universe.

Example 3. (Kluyver's Theorem) A justifiably famous result of Kluyver [Klu06] is that,

$$P_n(0;1) = \frac{1}{n+1}, \tag{9}$$

for $n \geqslant 2$. That is, after n unit steps in the plane, the probability of being within one unit of home is $1/(n+1)$.

- (9) is immediate from (8) on appealing to the Fundamental theorem of calculus since $J_1 = j_0'$. Amazing!
- How simple, how unexpected, how beautiful! An elementary proof of was given only recently [Ber13].

◇

4 The moment functions

To see a World in a Grain of Sand
And a Heaven in a Wild Flower,
Hold Infinity in the palm of your hand
And Eternity in an hour. – William Blake (1757–1827) in *Songs of Innocence.*

Theorem 1 gives the *moment function*

$$W_n(\nu; s) = \int_0^\infty x^s p_n(\nu; x) \mathrm{d}x$$

of the distance to the origin after n random steps as follows:

Theorem 4. (Bessel integral for the moments) *Let $n \geqslant 2$ and $d \geqslant 2$. For any $k \geqslant 0$,*

$$W_n(\nu; s) = \frac{2^{s-k+1}\Gamma\left(\frac{s}{2}+\nu+1\right)}{\Gamma(\nu+1)\Gamma\left(k-\frac{s}{2}\right)} \int_0^\infty x^{2k-s-1} \left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k j_\nu^n(x)\mathrm{d}x, \tag{10}$$

if $k - n(\nu + 1/2) < s < 2k$. In particular, for $n > 2$, the first pole of $W_n(\nu; s)$ occurs at $s = -(2\nu + 2) = -d$.

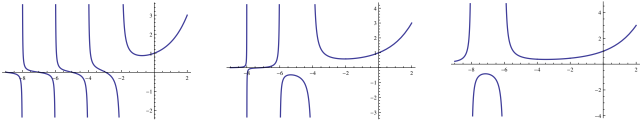
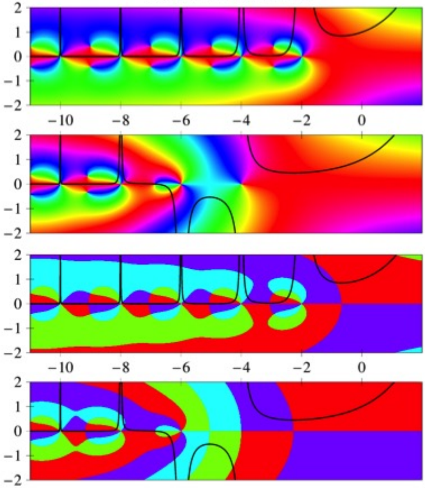


Figure 3: $W_3(\nu; s)$ on $[-9, 2]$ for $\nu = 0, 1, 2$.

Four step moments in two and four dimensions



- We next obtain from Theorem 1 an explicit combinatorial expression for the even moments.

Theorem 5. (Multinomial sum for moments) *The even moments of an n-step walk in dimension d are given by*

$$W_n(\nu; 2k) = \frac{(k+\nu)!\nu!^{n-1}}{(k+n\nu)!} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1,\ldots,k_n}\binom{k+n\nu}{k_1+\nu,\ldots,k_n+\nu}.$$

Example 6. With $k = 1$, Theorem 5 implies the second moment of an n-step random walk is $W_n(\nu; 2) = n$. Similarly, we find that

$$W_n(\nu; 4) = \frac{n(n(\nu + 2) - 1)}{\nu + 1}. \tag{11}$$

More generally, $W_n(\nu; 2k)$ is a polynomial of degree k in n, with coefficients that are rational in ν. For instance,

$$W_n(\nu; 6) = \frac{n(n^2(\nu + 2)(\nu + 3) - 3n(\nu + 3) + 4)}{(\nu + 1)^2} \tag{12}$$

and so on. Equation (12) shows that only in two or four dimensions can all the moments be integers. ◇

- Using the explicit expression for the even moments we derive the following convolution.

Corollary 7. (Moment recursion) *For positive integers* n_1, n_2, *half-integer* ν *and nonnegative integer* k *we have*

$$W_{n_1+n_2}(\nu; 2k) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)!(j+\nu)!} W_{n_1}(\nu; 2j) W_{n_2}(\nu; 2(k-j)). \tag{13}$$

- The case $n_2 = 1$, relates moments of an n-step walk to those of an $(n-1)$-step walk.

Example 8. (Integrality of two and four dimensional even moments) Corollary 7 is an efficient way to compute even moments in any dimension and so to data-mine. For illustration, because they are integral, we record the moments in two and four dimensions for $n = 2, 3, \ldots, 6$.

$$
\begin{aligned}
W_2(0;2k) &: \quad 1, 2, 6, 20, 70, 252, 924, 3432, 12870, \ldots \\
W_3(0;2k) &: \quad 1, 3, 15, 93, 639, 4653, 35169, 272835, 2157759, \ldots \\
W_4(0;2k) &: \quad 1, 4, 28, 256, 2716, 31504, 387136, 4951552, 65218204, \ldots \\
W_5(0;2k) &: \quad 1, 5, 45, 545, 7885, 127905, 2241225, 41467725, 798562125, \ldots \\
W_6(0;2k) &: \quad 1, 6, 66, 996, 18306, 384156, 8848236, 218040696, 5651108226, \ldots
\end{aligned}
$$

- For $n = 2$, these are central binomial coefficients, while, for $n = 3, 4$, these are Apéry-like sequences.

In general they are sums of squares of multinomial coefficients and so integers. Likewise, the initial even moments in four dimensions are as follows.

$$
\begin{aligned}
W_2(1;2k) &: \quad 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots \\
W_3(1;2k) &: \quad 1, 3, 12, 57, 303, 1743, 10629, 67791, 448023, \ldots \\
W_4(1;2k) &: \quad 1, 4, 22, 148, 1144, 9784, 90346, 885868, 9115276, \ldots \\
W_5(1;2k) &: \quad 1, 5, 35, 305, 3105, 35505, 444225, 5970725, 85068365, \ldots \\
W_6(1;2k) &: \quad 1, 6, 51, 546, 6906, 99156, 1573011, 27045906, 496875786, \ldots
\end{aligned}
$$

Observe that the first three terms in each case are as determined in Example 6.

In the two-step case in four dimensions, we find that the even moments are the *Catalan numbers* C_k, that is

$$W_2(1;2k) = \frac{(2k+2)!}{(k+1)!(k+2)!} = C_{k+1}, \quad C_k := \frac{1}{k+1}\binom{2k}{k}. \tag{14}$$

In 2 and 4 dim *only*, all even moments are integers.

- This is obvious for $d = 2$ in Theorem 5 which gives

$$W_n(0;2k) = \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1,\ldots,k_n}^2$$

and counts *abelian squares* [RS09].

On the other hand, to show that $W_n(1;2k)$ is always integral,we recursively apply (13) and note that it is known (see Example 9) that the factors

$$\binom{k}{j}\frac{(k+1)!}{(k-j+1)!(j+1)!} = \frac{1}{j+1}\binom{k}{j}\binom{k+1}{j} \tag{15}$$

are integers for all nonnegative j and k. The numbers (15) are known as *Narayana numbers* and occur in various counting problems; see, [Sta99, Problem 6.36]. ◇

- Integrality of 4-dimensional moments is a deeper – and so arguably more beautiful – fact we first discovered numerically before being led to the Narayana triangle:

Example 9. (Narayana numbers and triangle) The recursion for $W_n(\nu;2k)$ is equivalent to: *for given* ν*, let* $A(\nu)$ *be the infinite lower triangular matrix with entries*

$$A_{k,j}(\nu) = \binom{k}{j}\frac{(k+\nu)!\nu!}{(k-j+\nu)!(j+\nu)!} \qquad (16)$$

for row indices $k = 0,1,2,\ldots$ *and columns* $j = 0,1,2,\ldots$.

Then the row sums of $A(\nu)^n$ *are given by the moments* $W_{n+1}(\nu;2k)$, $k = 0,1,2,\ldots$.

- For instance, in the case $\nu = 1$:

$$A(1) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \\ 1 & 3 & 1 & 0 & \\ 1 & 6 & 6 & 1 & \\ \vdots & & & & \ddots \end{bmatrix}, \qquad A(1)^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & \\ 12 & 9 & 1 & 0 & \\ 57 & 72 & 18 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},$$

The row sums $1, 2, 5, 14, \ldots$ and $1, 4, 22, 148, \ldots$ correspond to $W_2(1; 2k)$ and $W_4(1; 2k)$ of Ex. 8.

The first column of $A(\nu)$ is all 1's, so $W_n(\nu; 2k)$ also give first column of $A(\nu)^n$. $A(1)$ is known as the *Narayana triangle* or *Catalan triangle* [Slo14, `A001263`].[2] ◇

[2] The OEIS is a mathematical bird guide. We see/hear something striking and our guide points out the species.

- What is beautiful is that we completely describe the even moments in four dimensions in terms of powers of one known combinatorial matrix and the ubiquitous Catalan numbers.
- We have reduced probabilistic and analytic objects to purely combinatorial roots.
- We have illustrated another source of beauty in maths.
  - As we peal away parts of the onion we often uncover unexpected complexity in seemingly simple or unexplored settings.

- Further study is rewarded by a level of simplicity yet to be found below. We are offered a glimpse of infinity in Blake's grain of sand.
- I am reminded that fame in art and in mathematics can be changeable.
  - Consider Blake before Northrop Frye's 1947 book *A fearful symmetry*.
  - Consider also the impact of performances by Mendelsohn (1829) and Gould (1955) on our reception of Bach who died in 1750.

> *The difficulty lies, not in the new ideas, but in escaping the old ones, which ramify, for those brought up as most of us have been, into every corner of our minds.* – John Maynard Keynes[a]
>
> [a] KE Drexler, *Engines of Creation: The Coming Era of Nanotechnology*, New York, 1987.

The next equation

$$W_3(\nu; 2k) = {}_3F_2\left({-k, -k-\nu, \nu+1/2 \atop \nu+1, 2\nu+1} \middle| 4 \right). \qquad (17)$$

gives a hypergeometric expression for the even moments of a 3-step random walk.

- We discovered numerically that, in the plane, the *real part* of (17) still evaluates odd moments [BNSW11]. (e.g., $W_3(0;1) = \mathrm{Re}(1.5745972 \pm 0.12602652i)$).

Odd moments are much harder to obtain; it was first proved based on this observation, that the average distance of a planar 3-step random walk is

$$W_3(0;1) = A + \frac{6}{\pi^2}\frac{1}{A} \approx 1.5746, \tag{18}$$

where

$$W_3(0;-1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\tfrac{1}{3}\right) =: A. \tag{19}$$

- This discovery for me felt precisely like Keats' description of "On first looking into Chapman's Homer."[3]

We then proved the transcendental nature of odd moments of 3-step walks in all even dimensions by showing they are rational linear combinations of A and $1/(\pi^2 A)$.

- We first met planar moments in the symbolically accessible form

$$W_n(0;s) = \int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi}\left|e^{\theta_1 i}+e^{\theta_2 i}+\cdots+e^{\theta_n i}\right|^s \mathrm{d}\theta_n\mathrm{d}\cdots\theta_2\mathrm{d}\theta_1. \tag{20}$$

[3] "Then felt I like some watcher of the skies
When a new planet swims into his ken;" see http://www.poetryfoundation.org/poem/173746.

5 Densities of 5-step walks

> *The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.* – G.H. Hardy (1887-1977)[a]
>
> [a] In his delightful *A Mathematician's Apology*, 1941.

Almost all mathematicians agree with Hardy – until asked to put flesh on the bones of his endorsement of beauty.

- Beauty may be the first test but it is the eye of the beholder.

- Hardy, in the twelfth of his twelve lectures given as a eulogy for the singular Indian genius Srinivasa Ramanujan (1887–1921), described a result of Ramanujan, now viewed as one his finest, somewhat dismissively as

  a remarkable formula with many parameters.[4]

[4] See Ole Warnaar's 2013 contribution in "Srinivasa Ramanujan Going Strong at 125, Part II," available at http://www.ams.org/notices/201301/rnoti-p10.pdf.

The 5-step densities for dimensions up to 9 are shown in Figure 4. A peculiar feature in the plane is the striking (approximate) linearity on the initial interval $[0, 1]$:

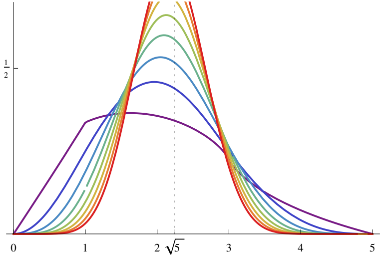


Figure 4: $p_5(\nu; x)$ for $\nu = 0, \frac{1}{2}, 1, \ldots, \frac{7}{2}$

As Pearson [Pea06] commented:

> *The graphical construction, however carefully reinvestigated, did not permit of our considering the curve to be anything but a* straight *line. . . Even if it is not absolutely true, it exemplifies the extraordinary power of such integrals of J products [that is, (5)] to give extremely close approximations to such simple forms as horizontal lines.*

- In 1963 Fettis [Fet63] established nonlinearity (via numerical estimation).

Rigorously, we proved that

$$p_5(0;x) = 0.329934\, x + 0.00661673\, x^3 + 0.000262333\, x^5 + \ldots,$$

which illustrates the near linearity of $p_5(0;x)$ for values of $x < 1$.

- Is this result beautiful because it entirely resolves the issue of whether the density is linear on $[0,1]$ or is it ugly because it demolishes the apparent linearity?

6 Conclusions

Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigour should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere.
–W.S. (Bill) Angelin.[a]

[a] From his article "Mathematics and History", *Mathematical Intelligencer*, vol. 14, no. 4, (1992), 6-12.

- The tidy ex post facto beauty of a well written mathematics textbook or Bourbaki monograph is quite different from the beauty of still wild parts of maths.

Remark 10. (Meditation on beauty, II) Bessel functions like some other special functions (e.g., the Gamma function, hypergeometric functions and elliptic integrals) are extraordinary in both their theoretical ubiquity and applicability.

Because of my (pure) maths training, I knew them only peripherally until my research moved into mathematical physics, random walk theory and other "boundary" fields.

They can even be used to produce immensely complicate standing water waves, spelling out corporate names! [5]

- To me now, they are a mathematical gem, every facet of which rewards further examination.
- To me as a student, they were only the solution to a second-order algebraic definition which I had to look up each time.
- Moreover, looking them up is now easy and fun thanks to sources like [DLMF].

[5] See http://www.openscience.org/blog/?p=193. (See also Pearson's comment in Section 5.)

I could make similar remarks about combinatorial objects, such as the Catalan numbers and the Narayana triangle.

- In this case familiarity breeds content not contempt.
- Moreover, computer algebra packages make it wonderfully easy to become familiar with the objects.

Remark 11. (Meditation on beauty, III) We have shown that quite delicate results are possible for densities and moments of walks in arbitrary dimensions, especially for two, three and four steps.

(a) We find it interesting that induction between dimensions provided methods to show results in the plane that we previously could not establish [BSWZ12].

(b) We also should emphasize the crucial role played by intensive computer experimentation and by computer algebra ('big data' meets modern computation).

(c) One stumbling block is that currently *Mathematica*, and to a lesser degree *Maple*, struggle with computing various Bessel integrals to more than a few digits — thus requiring considerable computational effort or ingenuity.

(d) The seemingly necessary interplay from combinatorial to analytic to probabilistic tools and back, is ultimately one of the greatest sources of pleasure and beauty of the work.

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