Experimental Mathematics:

Apéry-Like Identities for $\zeta(n)$

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We wish to consider one of the most fascinating and glamorous functions of analysis, the Riemann zeta function. (R. Bellman)

Siegel found several pages of ... numerical calculations with ... zeroes of the zeta function calculated to several decimal places each. As Andrew Granville has observed "So much for pure thought alone." (JB & DHB)

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AK Peters 2004 **Talk Revised**: 03–29–05

Apéry-Like Identities for $\zeta(n)$

The final lecture comprises a research level case study of generating functions for zeta functions. This lecture is based on past research with David Bradley and current research with David Bailey.

One example is

$$\mathcal{Z}(x) := 3\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2-x^2)} \prod_{n=1}^{k-1} \frac{4x^2-n^2}{x^2-n^2}$$

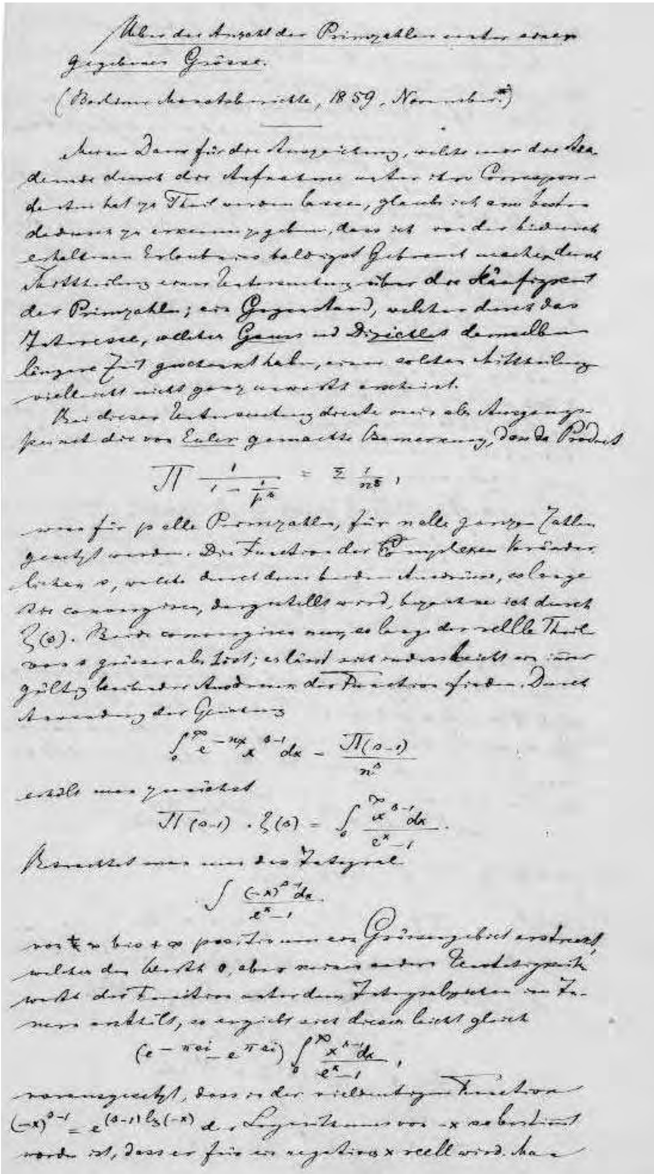
$$= \sum_{n=1}^{\infty} \frac{1}{n^2-x^2} \tag{1}$$

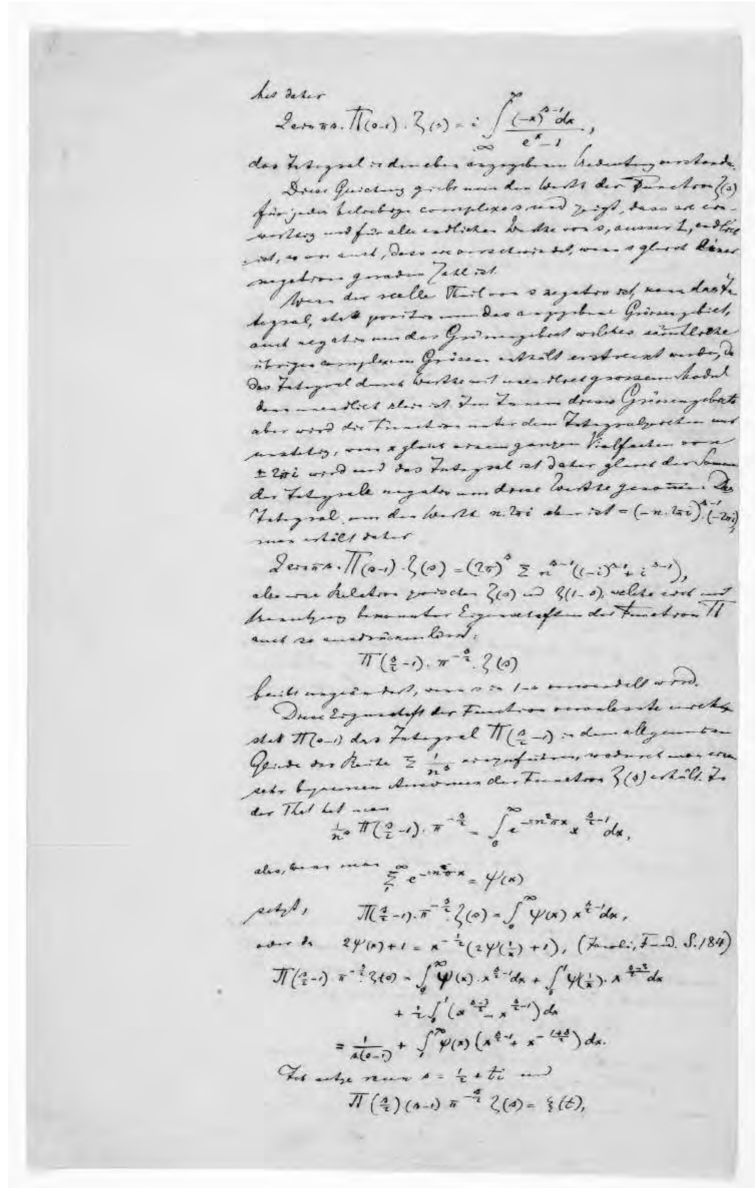
$$\left[= \sum_{k=0}^{\infty} \zeta(2k+2)\, x^{2\,k} = \frac{1-\pi x \cot(\pi x)}{2x^2}\right].$$

Note that with $x=0$ this recovers

$$3\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}k^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) \tag{2}$$

Riemann's Original 1859 Manuscript

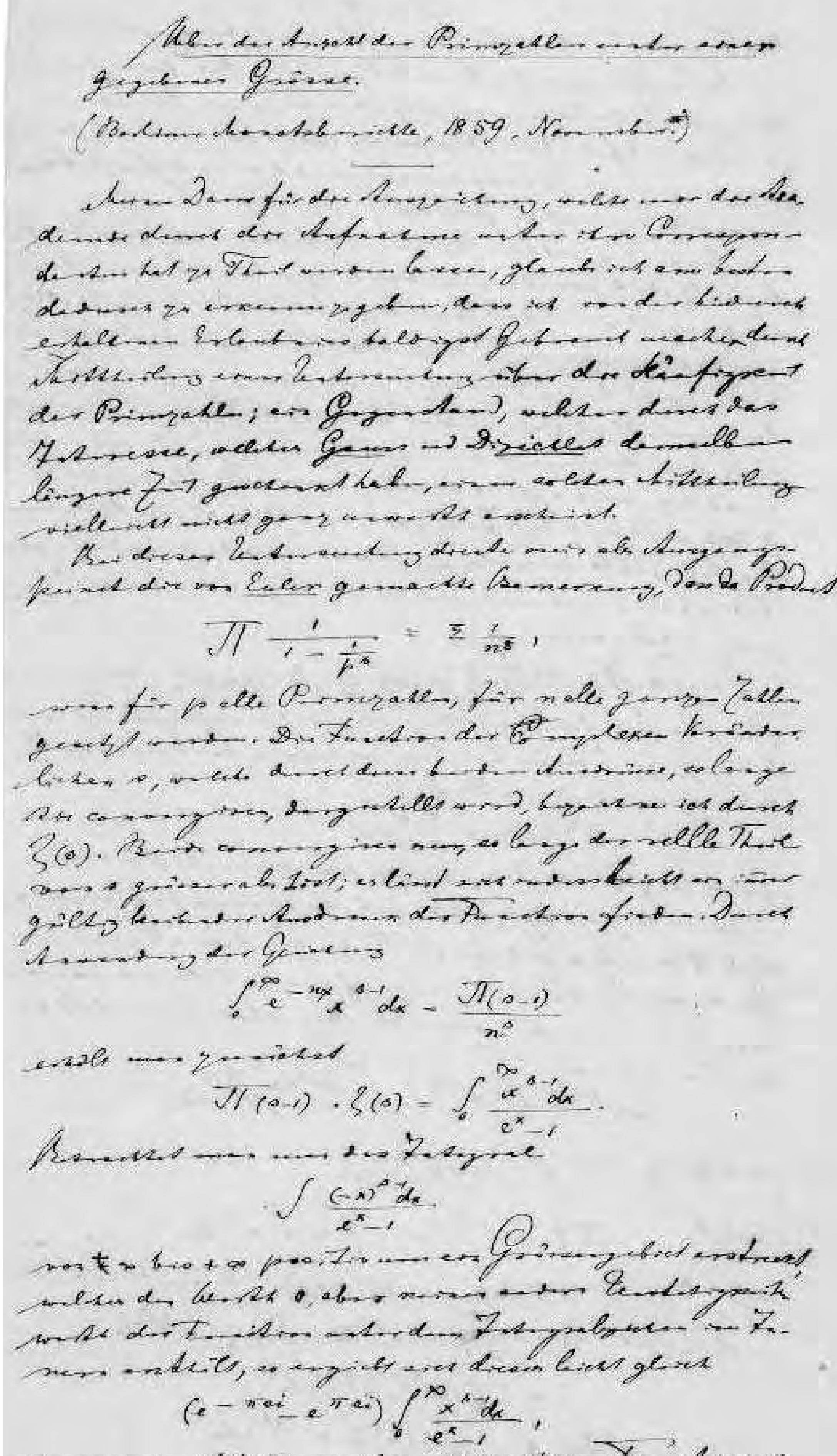




- Showing the *Euler product* and the *reflection formula* ($s \mapsto 1 - s$). Even the notation is as today.
  - As seen recently on **Numb3rs** and **Law and Order**—ζ is starting to compete with π.

George Friedrich Bernard Riemann

(1826-1866)



Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Berliner Monatsberichte, 1859, November.)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\,dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\cdot\zeta(s) = \int_0^\infty \frac{x^{s-1}\,dx}{e^x-1}.$$

Betrachtet man nun das Integral

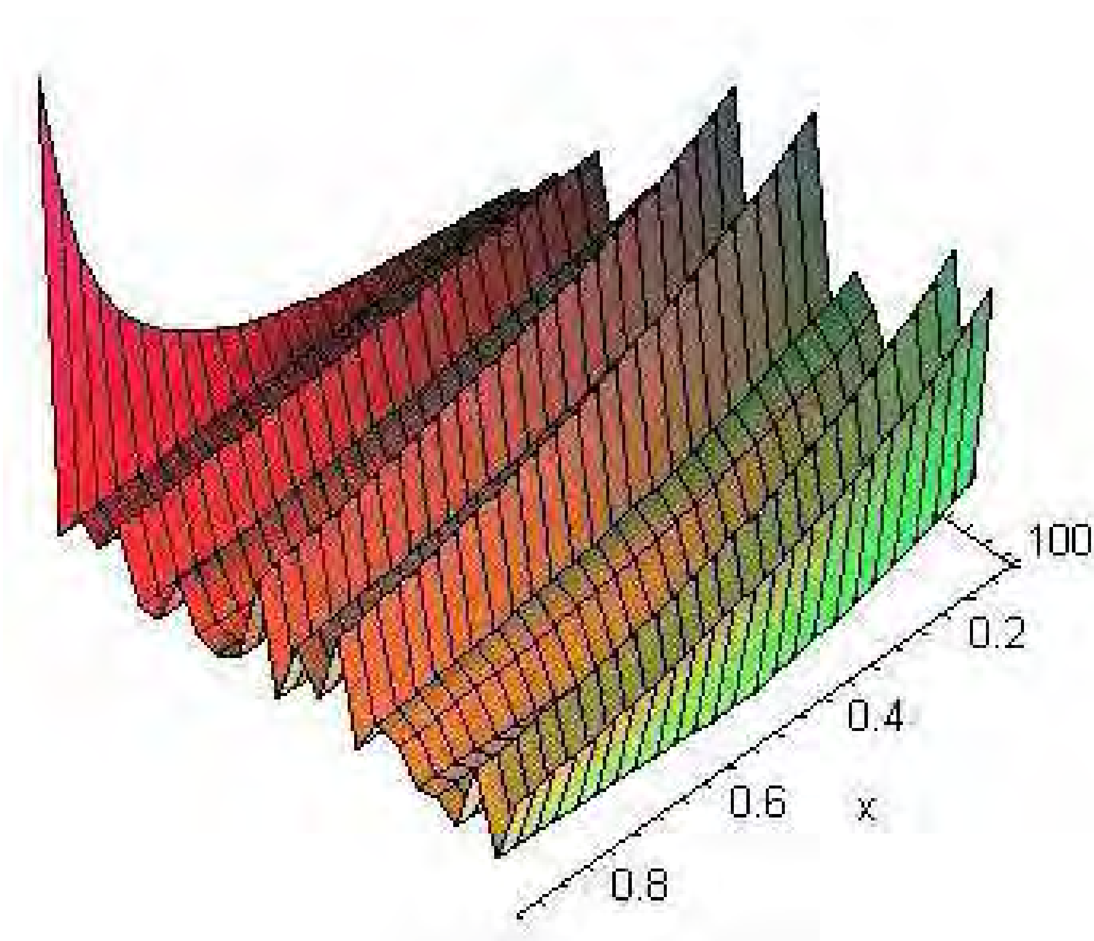
$$\int \frac{(-x)^{s-1}\,dx}{e^x-1}$$

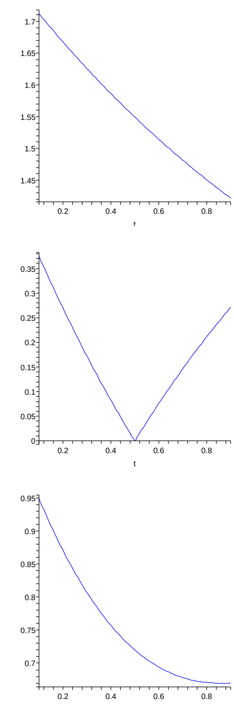
von $+\infty$ bis $+\infty$ positiv um ein Grössengebiet erstreckt, welches den Werth 0, aber keinen andern Unstetigkeitswerth der Function unter dem Integralzeichen im Innern enthält, so ergiebt sich dieses leicht gleich

$$(e^{-\pi si} - e^{\pi si})\int_0^\infty \frac{x^{s-1}\,dx}{e^x-1},$$

The Riemann Hypothesis

$\$ \vee £ \vee \ldots$ The only Millennium *and* Hilbert Problem





Curves at and around the 1st zero

. . . .

All non-real zeros have real part 'one half'

★★ Note the monotonicity of $x \mapsto |\zeta(x + iy)|$.

This is equivalent to (RH) as discovered in 2002*.

*By Zvengerowski and Saidal in a complex analysis class.

ODLYZKO and the NON-TRIVIAL ZEROS

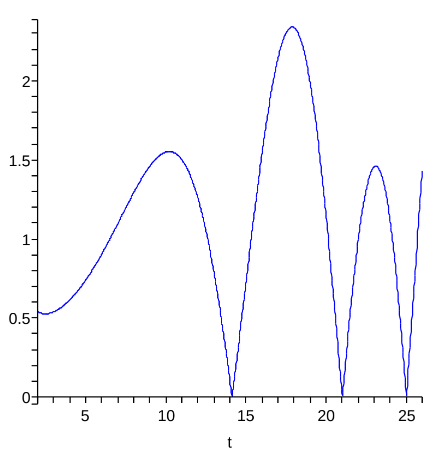
Andrew Odlyzko: Tables of zeros of the Riemann zeta function

- **The first 100,000 zeros of the Riemann zeta function, accurate to within 3*10^(-9).** [text, 1.8 MB] [gzip'd text, 730 KB]
- **The first 100 zeros of the Riemann zeta function, accurate to over 1000 decimal places.** [text]
- **Zeros number 10^12+1 through 10^12+10^4 of the Riemann zeta function.** [text]
- **Zeros number 10^21+1 through 10^21+10^4 of the Riemann zeta function.** [text]
- **Zeros number 10^22+1 through 10^22+10^4 of the Riemann zeta function.** [text]

Up [Return to home page]

14.134725142 21.022039639 **25.010857580** 30.424876126

32.935061588 **37.586178159** 40.918719012 **43.327073281**



▶ The imaginary parts of the first 8 zeroes; they do lie on the critical line.

▶ At 10^{22} the *Law of small numbers* still rules.

▶ Real zeroes are at $-2\mathbb{N}$

/www.dtc.umn.edu/∼odlyzko/

An ELEMENTARY WARMUP

The well known series for $\arcsin^2$ generalizes fully:

Theorem. For $|x| \le 2$ and $N = 1, 2, \ldots$

$$\frac{\arcsin^{2N}\left(\frac{x}{2}\right)}{(2N)!} = \sum_{k=1}^{\infty} \frac{H_N(k)}{\binom{2\,k}{k} k^2}\, x^{2\,k}, \tag{3}$$

where $H_1(k) = 1/4$ and

$$H_{N+1}(k) := \sum_{n_1=1}^{k-1} \frac{1}{(2n_1)^2} \sum_{n_2=1}^{n_1-1} \frac{1}{(2n_2)^2} \cdots \sum_{n_N=1}^{n_{N-1}-1} \frac{1}{(2n_N)^2},$$

and

$$\frac{\arcsin^{2N+1}\left(\frac{x}{2}\right)}{(2N+1)!} = \sum_{k=0}^{\infty} \frac{G_N(k)\binom{2\,k}{k}}{2(2k+1)4^{2k}} x^{2k+1}, \tag{4}$$

where $G_0(k) = 1$ and

$$G_N(k) := \sum_{n_1=0}^{k-1} \frac{1}{(2n_1+1)^2} \sum_{n_2=0}^{n_1-1} \frac{1}{(2n_2+1)^2} \cdots \sum_{n_N=0}^{n_{N-1}-1} \frac{1}{(2n_N+1)^2}.$$

▶ Thus, for $N = 1, 2, \ldots$ $\qquad$ [$N = 1$ recovers (2)]

$$\sum_{k=1}^{\infty} \frac{H_N(k)}{\binom{2\,k}{k} k^2} = \frac{\pi^{2N}}{6^{2N}\,(2N)!}.$$

$$\left[\frac{1}{72}\,\pi^2, \frac{1}{31104}\,\pi^4, \frac{1}{33592320}\,\pi^6, \frac{1}{67722117120}\,\pi^8\right]$$

BINOMIAL SUMS and PSLQ

▶ Any relatively prime integers p and q such that

$$\zeta(5) \stackrel{?}{=} \frac{p}{q} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}}$$

have q astronomically large (as "lattice basis reduction" shows).

▶ But $\cdots$ PSLQ yields in *polylogarithms*:

$$\begin{aligned} A_5 &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}} = 2\zeta(5) \\ &- \frac{4}{3}L^5 + \frac{8}{3}L^3\zeta(2) + 4L^2\zeta(3) \\ &+ 80 \sum_{n>0} \left(\frac{1}{(2n)^5} - \frac{L}{(2n)^4} \right) \rho^{2n} \end{aligned}$$

where

$$L := \log(\rho)$$

and

$$\rho := (\sqrt{5} - 1)/2$$

with similar formulae for A_4, A_6, S_5, S_6 and S_7.

- A less known formula for $\zeta(5)$ due to Koecher suggested generalizations for $\zeta(7), \zeta(9), \zeta(11) \ldots$

- Again the coefficients were found by integer relation algorithms. *Bootstrapping* the earlier pattern kept the search space of manageable size.

- For example, and simpler than Koecher:

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}} + \frac{25}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4} \qquad (5)$$

▶ We were able – by finding integer relations for $n = 1, 2, \ldots, 10$ – to encapsulate the formulae for $\zeta(4n+3)$ in a single conjectured generating function, (entirely *ex machina).*

▶ The discovery was:

Theorem 1 *For any complex* z,

$$\sum_{n=0}^{\infty} \zeta(4n+3) z^{4n} = \sum_{k=1}^{\infty} \frac{1}{k^3(1 - z^4/k^4)} \tag{6}$$

$$= \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k} (1 - z^4/k^4)} \prod_{m=1}^{k-1} \frac{1 + 4z^4/m^4}{1 - z^4/m^4}.$$

- The first '=' is easy. The second is quite unexpected in its form.

- Setting $z = 0$ yields Apéry's formula for $\zeta(3)$ and the coefficient of z^4 is (14).

$$\boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k \binom{2k}{k}} = \frac{2}{\sqrt{5}} \log\left(\frac{1+\sqrt{5}}{2}\right)} \tag{7}$$

HOW IT WAS FOUND

- The first ten cases show (6) has the form

$$\frac{5}{2}\sum_{k\geq 1}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\frac{P_k(z)}{(1-z^4/k^4)}$$

for *undetermined* P_k; with abundant data to compute

$$\boxed{P_k(z)=\prod_{m=1}^{k-1}\frac{1+4z^4/m^4}{1-z^4/m^4}.}$$

- We found many reformulations of (6), including a marvellous **finite** sum:

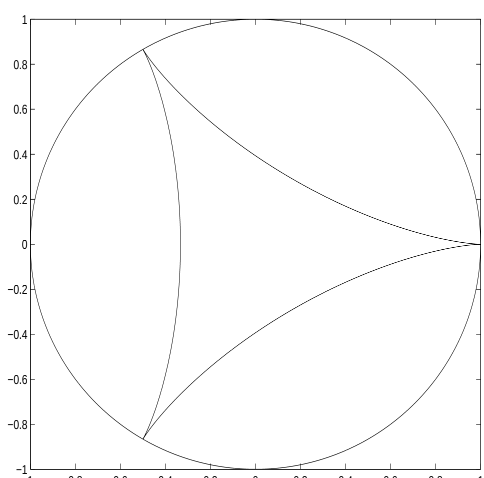
$$\boxed{\sum_{k=1}^{n}\frac{2n^2}{k^2}\frac{\prod_{i=1}^{n-1}(4k^4+i^4)}{\prod_{i=1,\,i\neq k}^{n}(k^4-i^4)}=\binom{2n}{n}. \qquad (8)}$$

- Obtained via Gosper's (Wilf-Zeilberger type) *telescoping algorithm* after a mistake in an electronic Petri dish ('infty' $\neq$ 'infinity').

- ► This finite identity was subsequently proved by Almkvist and Granville (*Experimental Math*, 1999) thus finishing the proof of (6) and giving a rapidly converging series for any $\zeta(4N+3)$ where N is positive integer.

★ Perhaps shedding light on the irrationality of $\zeta(7)$?

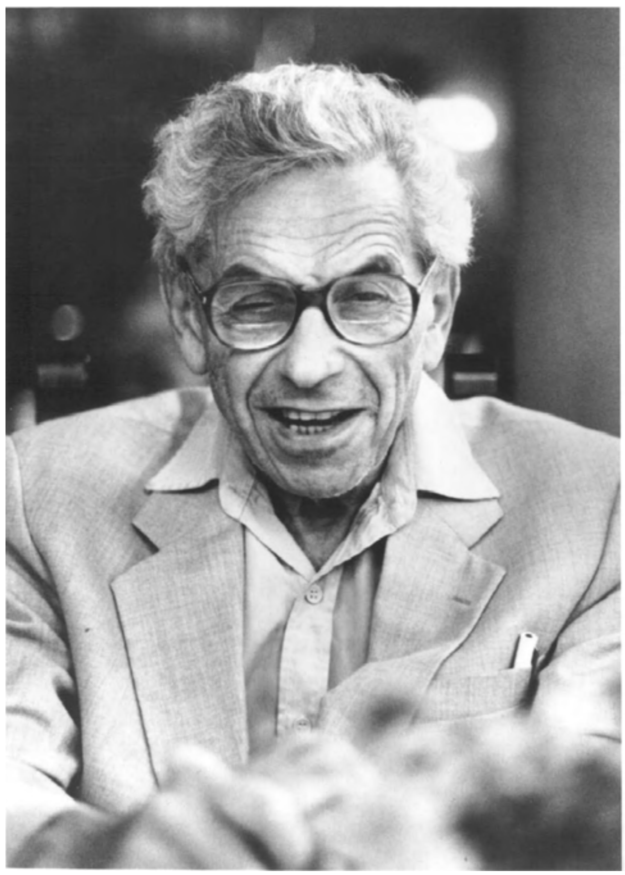
Recall that $\zeta(2N+1)$ is not proven irrational for $N > 1$. One of $\zeta(2n+3)$ for $n = 1, 2, 3, 4$ is irrational (Rivoal et al).



Kakeya's needle was an excellent false conjecture

PAUL ERDŐS (1913-1996)

Paul Erdős, when shown (8) shortly before his death, rushed off.



Twenty minutes later he returned saying he did not know how to prove it but if proven it would have implications for Apéry's result ('$\zeta(3)$ is irrational').

The CURRENT RESEARCH

- We now document the discovery of two generating functions for $\zeta(2n+2)$, analogous to earlier work for $\zeta(2n+1)$ and $\zeta(4n+3)$, initiated by Koecher and completed by various other authors.

Recall: an *integer relation relation algorithm* is an algorithm that, given n real numbers $(x_1, x_2, \cdots, x_n)$, finds integers a_i such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0,$$

at least to within available numerical precision, or else establishes that there are no integers a_i within a ball of radius A—in the Euclidean norm:

$$A = (a_1^2 + a_2^2 + \cdots + a_n^2)^{1/2}.$$

- Helaman Ferguson's "PSLQ" is the most widely known integer relation algorithm, although variants of the "LLL" algorithm are also well used.

© Such algorithms are now the basis of the the "Recognize" function in *Mathematica* and of the "identify" function in *Maple*, and form the basis of our work.

- The existence of series formulas involving central binomial coefficients in the denominators for the $\zeta(2)$, $\zeta(3)$, and $\zeta(4)$—and the role of the formula for $\zeta(3)$ in Apéry's proof of its irrationality —has prompted considerable effort to extend these results to larger integer arguments.

The formulas in question are

$$\zeta(2) = 3\sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}}, \tag{9}$$

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}, \tag{10}$$

$$\zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}. \tag{11}$$

(9) has been known since the 19C—it relates to $\arcsin^2(x)$—while (10) was variously discovered in the 20C and (11) was proved by Comptet. These three are the only single term identities or "*seeds*".

- A coherent proof of all three was provided by Borwein-Broadhurst-Kamnitzer in course of a more general study of such central binomial series and so called *multi-Clausen sums*.

These results make it tempting to conjecture

$$\mathfrak{Q}_5 = \zeta(5)/\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^5\binom{2k}{k}}$$

is a simple rational or algebraic number.

Example. *Integer relation shed light on* $\mathfrak{Q}_5$.

1997 If $\mathfrak{Q}_5$ is algebraic of degree 24 then the Euclidean norm of coefficients exceeds 2×10^{37}.

2005 Using 10,000-digit precision, the norm exceeds 1.24×10^{383}.

2005 If $\zeta(5)$ is algebraic of degree 24 its norm exceeds 1.98×10^{380}. □

Moreover, a study of *polylogarithmic ladders in the golden ratio* (BBK), produced

$$2\,\zeta(5) - \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}} = \frac{5}{2}\,\mathrm{Li}_5(\rho) - \frac{5}{2}\,\mathrm{Li}_4(\rho)\ln\rho + \zeta(3)\log^2\rho - \frac{1}{3}\zeta(2)\log^3\rho - \frac{1}{24}\log^5\rho, \qquad (12)$$

where $\rho = (3-\sqrt{5})/2$ and where $\mathrm{Li}_N(z) = \sum_{k=1}^{\infty} z^k/k^N$ is the *polylogarithm* of order N.

- Since the terms on the right hand side are almost certainly algebraically independent, we see how unlikely it is that $\mathcal{Q}_5$ is rational.

- We note that at present it is proven only that one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational; and that a nontrivial density of all odd values is.

Given the negative result from PSLQ computations for $\mathfrak{Q}_5$, Bradley and JMB systematically investigated the possibility of a multi-term identity of this general form for $\zeta(2n+1)$.

The following was then recovered early in experimental searches using computer-based integer relation tools:

$$\zeta(5) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5\binom{2k}{k}} - \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^2} \quad (13)$$

- ▶ In a similar way, identities were found for $\zeta(7), \zeta(9)$ and $\zeta(11)$ (the identity for $\zeta(9)$ is listed later):

$$\zeta(7) = \frac{\mathbf{5}}{\mathbf{2}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}} + \frac{\mathbf{25}}{\mathbf{2}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4} \tag{14}$$

$$\begin{aligned}\zeta(11) &= \frac{\mathbf{5}}{\mathbf{2}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{11}\binom{2k}{k}} + \frac{\mathbf{25}}{\mathbf{2}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4} \\ &- \frac{75}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^8} \\ &+ \frac{125}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\sum_{i=1}^{k-1}\frac{1}{i^4}.\end{aligned} \tag{15}$$

- Note that the formulas for $\zeta(7)$ and $\zeta(11)$ include, as the first term, a close analogue of the formula for $\zeta(3)$ given above, and the first two coefficients in (15) clearly repeat those in (14).
  - this suggested that a "bootstrap" approach might allow production of enough higher-level formulas for $\zeta(4n+3)$ for $m = 2, 3, \cdots$ to determine the closed form:

- Indeed, this was the case; in fact, such "bootstrapping" helped by restricting the number of multiple relations that otherwise makes the analysis difficult or impossible.

  - we were able to sum all higher variables up to $k-1$ which significantly speeds up numerical computation

- such issues have, so far, prevented the generalization of formulas such as the one above for $\zeta(5)$ to the general case of $\zeta(4n+1)$

The following general formula, due to Koecher following techniques of Knopp and Schur,

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\binom{2\,k}{k}\,k^3}\frac{5\,k^2-x^2}{k^2-x^2}\prod_{n=1}^{k-1}\left(1-\frac{x^2}{n^2}\right)$$
$$= \sum_{n=1}^{\infty}\frac{1}{n\left(n^2-x^2\right)}. \tag{16}$$

gives (13) as its second term but more complicated expressions for $\zeta(7)$ and $\zeta(11)$ than (14) and (15).

After bootstrapping, an application of the "Pade" function, which in both *Mathematica* and *Maple* produces Padé approximations to a rational function satisfied by a truncated power series, produced the following remarkable result:

$$\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}(1-x^4/k^4)}\prod_{m=1}^{k-1}\left(\frac{1+4x^4/m^4}{1-x^4/m^4}\right)$$

$$= \sum_{n=0}^{\infty}\zeta(4n+3)x^{4n} = \sum_{k=1}^{\infty}\frac{1}{k^3(1-x^4/k^4)} \qquad (17)$$

- rigorously established by Almkvist-Granville, it can now be handled in part symbolically by Wilf-Zeilberger (WZ) methods

It is also the $x=0$ case of the unified formula *conjectured by Cohen after much experiment* (Rivoal, 2005):

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k\binom{2\,k}{k}}\frac{5\,k^2-x^2}{k^4-x^2k^2-y^4} \times \prod_{n=1}^{k-1}\frac{\left(n^2-x^2\right)^2+4\,y^4}{n^4-x^2n^2-y^4}$$

$$= \sum_{n=1}^{\infty}\frac{n}{n^4-x^2n^2-y^4} \qquad (18)$$

in which setting $y=0$ recovers (16).

- Stimulated by Rivoal's paper, we decided to revisit the even ζ-values.

An analogous, but more deliberate, experimental procedure, as detailed below yielded a formula for $\zeta(2n+2)$ that is pleasingly parallel to (17):

$$\begin{aligned} & 3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}\left(1-x^2/k^2\right)}\prod_{m=1}^{k-1}\left(\frac{1-4x^2/m^2}{1-x^2/m^2}\right) \\ = & \sum_{n=0}^{\infty}\zeta(2n+2)\,x^{2n}=\sum_{m=1}^{\infty}\frac{1}{\left(m^2-x^2\right)} \qquad (19) \\ = & \frac{\pi\cot(\pi x)\,x-1}{x^2}. \end{aligned}$$

OCR and Touch

▷ We finish by discussing the existence of a formula based on the seed $\zeta(4)$, and like questions.

The Details for $\zeta(2n+2)$

▷ We applied a similar though distinct experimental approach to produce a generating function for $\zeta(2n+2)$. We describe this process of discovery in some detail as the general technique appears to be quite fruitful.

Conjecture: $\zeta(2n+2)$ is a rational combination of terms of the form

$$\sigma(2r;[2a_1,\cdots,2a_N]) := \sum_{k>n_i>0} \frac{1}{k^{2r}\binom{2k}{k}\prod_{i=1}^{N} n_i^{2a_i}}, \tag{20}$$

where $r+\sum_{i=1}^{N} a_i = n+1$, and the a_i are listed in nonincreasing order

- RHS is independent of the order of the a_i

One can then write

$$\begin{aligned} Z(x) &:= \sum_{n=0}^{\infty} \zeta(2n+2)\, x^{2n} \qquad (21) \\ &= \sum_{n=0}^{\infty}\sum_{r=1}^{\infty}\sum_{\pi\in\Pi(n-r)} \alpha(k,\pi)\,\sigma(2r;2\pi)x^{2r+2(n-r)}, \end{aligned}$$

as $\Pi(m)$ ranges over *additive partitions* of m.

Write $\alpha(\pi) := \alpha(0,\pi)$ and define $\widehat{\sigma}_k([\cdot]) := 1$ for the null partition $[\cdot]$, and, for a partition $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$ of $m > 0$, written in nonincreasing order,

$$\widehat{\sigma}_k(\pi) := \sum_{k>n_i>0} \frac{1}{n_i^{2\pi_1} \cdots n_N^{2\pi_N}}. \tag{22}$$

- The α's *appear to be* independent of k:

$$\begin{aligned} Z(x) &= \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \sum_{\pi \in \Pi(n-r)} \alpha(\pi)\, \sigma(2r; 2\pi) x^{2r+2(n-r)} \\ &= \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}} \sum_{r=0}^{\infty} \frac{x^{2r}}{k^{2r+2}} \sum_{m=0}^{n-1} \sum_{\Pi(m)} \alpha(\pi) \widehat{\sigma}_k(\pi) x^{2m} \\ &= \sum_{k \geq 1} \frac{1}{\binom{2k}{k}(k^2 - x^2)} P_k(x) \end{aligned}$$

for functions $P_1, P_2, \ldots, P_k, \ldots$ whose form must be determined.

- Crucially we compute that for some $\gamma_{k,m}$

$$\begin{aligned} P_k(x) &= \sum_{m \geq 0} \gamma_{k,m}\, x^{2m} \\ &= \sum_{m=0}^{\infty} \left\{ \sum_{\pi \in \Pi(m)} \alpha(\pi) \sum_{n_i < k} \frac{1}{n_i^{2\pi_1} \cdots n_N^{2\pi_N}} \right\} x^{2m} \end{aligned} \tag{23}$$

★ **Our strategy** is to compute the first few explicit cases of $P_k(x)$, and hope they permit us to *extrapolate* the closed form, much as in the case of $\zeta(4n+3)$.

- Some examples we produced are shown below. At each step we "bootstrapped," noting that *certain* coefficients of the current result are the coefficients of the previous result.
  - we found the remaining coefficients by integer relation computations

- In particular, we computed high-precision (200-digit) numerical values of the assumed terms and the left-hand-side zeta value, and then applied PSLQ to find the rational coefficients.
  - in each case we "hard-wired" the first few coefficients to agree with the coefficients of the preceding formula

- Note below that in the sigma notation, the first few coefficients of each expression are simply the previous step's terms, *where the first argument of σ (corresponding to r) has been increased by two.*

- These terms (with coefficients in bold) are followed by terms for the other partitions

  - with all terms ordered lexicographically by partition

  - shorter partitions are listed before longer partitions, and, within a partition of a given length, larger entries are listed before smaller entries in the first position where they differ (the integers in brackets are nonincreasing):

$$
\begin{aligned}
\zeta(2) &= 3\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}k^2} \;=\; 3\sigma(2,[0]),\\
\zeta(4) &= \mathbf{3}\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}k^4} - 9\sum_{k=1}^{\infty}\frac{\sum_{j=1}^{k-1} j^{-2}}{\binom{2k}{k}k^2} \;=\; \mathbf{3}\sigma(4,[0]) - 9\sigma(2,[2])\\
\zeta(6) &= \mathbf{3}\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}k^6} - \mathbf{9}\sum_{k=1}^{\infty}\frac{\sum_{j=1}^{k-1} j^{-2}}{\binom{2k}{k}k^4} - \frac{45}{2}\sum_{k=1}^{\infty}\frac{\sum_{j=1}^{k-1} j^{-4}}{\binom{2k}{k}k^2}\\
&\quad + \frac{27}{2}\sum_{k=1}^{\infty}\sum_{j=1}^{k-1}\frac{\sum_{i=1}^{k-1} i^{-2}}{j^2\binom{2k}{k}k^2},\\
&= \mathbf{3}\sigma(6,[]) - \mathbf{9}\sigma(4,[2]) - \frac{45}{2}\sigma(2,[4]) + \frac{27}{2}\sigma(2,[2,2])\\
\zeta(8) &= \mathbf{3}\sigma(8,[]) - \mathbf{9}\sigma(6,[2]) - \frac{\mathbf{45}}{\mathbf{2}}\sigma(4,[4]) + \frac{\mathbf{27}}{\mathbf{2}}\sigma(4,[2,2])\\
&\quad -63\sigma(2,[6]) + \frac{135}{2}\sigma(2,[4,2]) - \frac{27}{2}\sigma(2,[2,2,2])\\
\zeta(10) &= \mathbf{3}\sigma(10,[]) - \mathbf{9}\sigma(8,[2]) - \frac{\mathbf{45}}{\mathbf{2}}\sigma(6,[4]) + \frac{\mathbf{27}}{\mathbf{2}}\sigma(6,[2,2])\\
&\quad -\mathbf{63}\sigma(4,[6]) + \frac{\mathbf{135}}{\mathbf{2}}\sigma(4,[4,2]) - \frac{\mathbf{27}}{\mathbf{2}}\sigma(4,[2,2,2])\\
&\quad -\frac{765}{4}\sigma(2,[8]) + 189\sigma(2,[6,2]) + \frac{675}{8}\sigma(2,[4,4])\\
&\quad -\frac{405}{4}\sigma(2,[4,2,2]) + \frac{81}{8}\sigma(2,[2,2,2,2]),
\end{aligned}
$$

- From the above results, one can immediately read that $\alpha([\cdot]) = 3$, $\alpha([1]) = -9$, $\alpha([2]) = -45/2$, $\alpha([1,1]) = 27/2$, and so forth.

Table 1 presents the values of α that we obtained in this manner.

Partition	α	Partition	α	Partition	α
[empty]	3/1	1	-9/1	2	-45/2
1,1	27/2	3	-63/1	2,1	135/2
1,1,1	-27/2	4	-765/4	3,1	189/1
2,2	675/8	2,1,1	-405/4	1,1,1,1	81/8
5	-3069/5	4,1	2295/4	3,2	945/2
3,1,1	-567/2	2,2,1	-2025/8	2,1,1,1	405/4
1,1,1,1,1	-243/40	6	-4095/2	5,1	9207/5
4,2	11475/8	4,1,1	-6885/8	3,3	1323/2
3,2,1	-2835/2	3,1,1,1	567/2	2,2,2	-3375/16
2,2,1,1	6075/16	2,1,1,1,1	-1215/16	1 ... 1	243/80
7	-49149/7	6,1	49140/8	5,2	36828/8
5,1,1	-27621/10	4,3	32130/8	4,2,1	-34425/8
4,1,1,1	6885/8	3,3,1	-15876/8	3,2,2	-14175/8
3,2,1,1	17010/8	3,1,1,1,1	-1701/8	2,2,2,1	10125/16
2,2,1,1,1	-6075/16	2,1,1,1,1,1	729/16	1 ... 1	-729/560
8	-1376235/56	7,1	1179576/56	6,2	859950/56
6,1,1	-515970/56	5,3	902286/70	5,2,1	-773388/56
5,1,1,1	193347/70	4,4	390150/64	4,3,1	-674730/56
4,2,2	-344250/64	4,2,1,1	413100/64	4,1,1,1,1	-41310/64
3,3,2	-277830/56	3,3,1,1	166698/56	3,2,2,1	297675/56
3,2,1,1,1	-119070/56	3,1,1,1,1,1	10206/80	2,2,2,2	50625/128
2,2,2,1,1	-60750/64	2,2,1,1,1,1	18225/64	2,1 ... 1	-1458/64
1 ... 1	2187/4480				

Alpha coefficients found by PSLQ

- Using these results, we use formula (23) to calculate series approximations—to order 17— for the functions $P_k(x)$:

$$
\begin{aligned}
P_3(x) &\approx 3-\frac{45}{4}x^2-\frac{45}{16}x^4-\frac{45}{64}x^6-\frac{45}{256}x^8-\frac{45}{1024}x^{10}-\frac{45}{4096}x^{12}-\frac{45}{16384}x^{14}\\
&\quad -\frac{45}{65536}x^{16}\\
P_4(x) &\approx 3-\frac{49}{4}x^2+\frac{119}{144}x^4+\frac{3311}{5184}x^4+\frac{38759}{186624}x^6+\frac{384671}{6718464}x^8\\
&\quad +\frac{3605399}{241864704}x^{10}+\frac{33022031}{8707129344}x^{12}+\frac{299492039}{313456656384}x^{14}\\
P_5(x) &\approx 3-\frac{205}{16}x^2+\frac{7115}{2304}x^4+\frac{207395}{331776}x^6+\frac{4160315}{47775744}x^8+\frac{74142995}{6879707136}x^{10}\\
&\quad +\frac{1254489515}{990677827584}x^{12}+\frac{20685646595}{142657607172096}x^{14}\\
&\quad +\frac{336494674715}{20542695432781824}x^{16}\\
P_6(x) &\approx 3-\frac{5269}{400}x^2+\frac{6640139}{1440000}x^4+\frac{1635326891}{5184000000}x^6-\frac{5944880821}{18662400000000}x^8\\
&\quad -\frac{212874252291349}{671846400000000}x^{10}-\frac{141436384956907381}{24186470400000000000}x^{12}\\
&\quad -\frac{70524260274859115989}{870712934400000000000000}x^{14}\\
&\quad -\frac{31533457168819214655541}{313456656384000000000000000}x^{16}\\
P_7(x) &\approx 3-\frac{5369}{400}x^2+\frac{8210839}{1440000}x^4-\frac{199644809}{5184000000}x^6-\frac{680040118121}{18662400000000}x^8\\
&\quad -\frac{278500311775049}{671846400000000}x^{10}-\frac{84136715217872681}{24186470400000000000}x^{12}\\
&\quad -\frac{22363377813883431689}{870712934400000000000000}x^{14}\\
&\quad -\frac{5560090840263911428841}{313456656384000000000000000}x^{16}.
\end{aligned}
$$

- With these approximations in hand, we attempt to determine closed-form expressions for $P_k(x)$.

  This can be done by using either "*Pade*" function in either *Mathematica* or *Maple*.

We obtained the following values*:

$$\begin{aligned}
P_1(x) &= 3 \\
P_2(x) &= \frac{3(4x^2-1)}{(x^2-1)} \\
P_3(x) &= \frac{12(4x^2-1)}{(x^2-4)} \\
P_4(x) &= \frac{12(4x^2-1)(4x^2-9)}{(x^2-4)(x^2-9)} \\
P_5(x) &= \frac{48(4x^2-1)(4x^2-9)}{(x^2-9)(x^2-16)} \\
P_6(x) &= \frac{48(4x^2-1)(4x^2-9)(4x^2-25)}{(x^2-9)(x^2-16)(x^2-25)} \\
P_7(x) &= \frac{192(4x^2-1)(4x^2-9)(4x^2-25)}{(x^2-16)(x^2-25)(x^2-36)}
\end{aligned}$$

♦ These results immediately *predict* the general form of a generating function identity:

*A bug in first alpha run gave a more complicated numerator for P_5 !

$$\mathcal{Z}(x) = 3\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2-x^2)} \prod_{n=1}^{k-1} \frac{4x^2-n^2}{x^2-n^2} \quad (24)$$

$$= \sum_{k=0}^{\infty} \zeta(2k+2)x^{2k} = \sum_{n=1}^{\infty} \frac{1}{n^2-x^2}$$

$$= \frac{1-\pi x \cot(\pi x)}{2x^2} \quad (25)$$

We have confirmed this result in several ways:

1. *Symbolically computing the series* coefficients of the LHS and the RHS of (25), and have verified that they agree up to the term with x^{100}.

2. We *verified* that $\mathcal{Z}(1/6)$, computing using (24), *agrees with* $18-3\sqrt{3}\pi$, computed using (25), to over 2,500 digit precision; likewise for $\mathcal{Z}(1/2) = 2$, $\mathcal{Z}(1/3) = 9/2 - 3\pi/(2\sqrt{3})$, $\mathcal{Z}(1/4) = 8 - 2\pi$ and $\mathcal{Z}(1/\sqrt{2}) = 1 - \pi/\sqrt{2} \cdot \cot(\pi/\sqrt{2})$.

3. We then *checked* that formula (24) gives the same numerical value as (25) for the 100 *pseudo-random values* $\{m\pi\}$, for $1 \le m \le 100$, where $\{\cdot\}$ denotes fractional part.

A Computational Proof

- Identity (24)–(25) can be proven by the methods of Rivoal's recent paper, which combine those in Borwein-Bradley and Almkvist-Granville. This relies on the *equivalent* finite identity:

$$\boxed{3n^2 \sum_{k=n+1}^{2n} \frac{\prod_{m=n+1}^{k-1} \frac{4n^2-m^2}{n^2-m^2}}{\binom{2k}{k}\left(k^2-n^2\right)} = \frac{1}{\binom{2n}{n}} - \frac{1}{\binom{3n}{n}}}$$

  – we rewrite (26) as

$${}_3F_2\left(\begin{matrix}3n, n+1, -n \\ 2n+1, n+1/2\end{matrix}; \frac{1}{4}\right) = \frac{\binom{2n}{n}}{\binom{3n}{n}}. \qquad (26)$$

  and set $P(n) = {}_3F_2\left(\begin{matrix}3n,n+1,-n \\ 2n+1,n+1/2\end{matrix}; \frac{1}{4}\right)$, $R(n) = \binom{2n}{n}/\binom{3n}{n}$. Then $P(0) = 1 = R(0)$ while

$$\frac{P(n+1)}{P(n)} = \frac{4\,(2n+1)^2}{3\,(3n+2)(3n+1)} = \frac{R(n+1)}{R(n)},$$

  where *Maple* or WZ gives the simplification.

  – thus, *inductively* $P(n) = R(n)$ for all n.

- We have proven (26). **QED**

The Details for $\zeta(2n+4)$

We have likewise now obtained for the *third seed*:

$$\zeta(4) = \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}},$$

the generating function

$$\begin{aligned} \mathcal{W}(x) &= \sum_{k=1}^{\infty} \frac{1}{\binom{2\,k}{k} k^2} \frac{1}{k^2 - x^2} \prod_{n=1}^{k-1} \left(1 - \frac{x^2}{n^2}\right) \\ &= \sum_{k=1}^{\infty} \frac{1}{(2k)!} \frac{\prod_{n=1}^{k-1} \left(n^2 - x^2\right)}{k^2 - x^2} \qquad (27) \\ &= \sum_{n=0}^{\infty} \gamma_n \, \zeta(2n+4) \, x^{2n} \qquad (28) \\ &\stackrel{?}{=} \alpha_0 \sum_{n=1}^{\infty} \frac{1}{n^4} \mathcal{R}\left(\frac{x^2}{n^2}\right) \qquad (29) \end{aligned}$$

where the coefficients γ_n are again computable rational numbers:

$$\begin{aligned}\mathcal{W}(x) &= \frac{17}{36}\zeta(4)+\frac{313}{648}\zeta(6)x^2+\frac{23147}{46656}\zeta(8)x^4\\ &+ \frac{1047709}{2099520}\zeta(10)x^6+O\left(x^8\right).\end{aligned}$$

- We observe that for integers, η_{2n},

$$\gamma_{2n}=\frac{\eta_{2n}}{6^{2n-2}\mathrm{numer}(\mathrm{B_{2n}})}.$$

  - this suggest that *perhaps we are looking at multiples of* arcsin(1/2) not Zeta values.

    Indeed,

$$\sigma(2;[\underbrace{2,\cdots,2}_{N-1}])=\frac{(\pi/3)^{2N}}{(2N)!},$$

    for $N\geq 1$.

- The η_{2n} values begin

$$17, 626, 23147, 4190836, 20880863207\ldots$$

We aim so to determine the form of the function $\mathcal{R}$. The anticipated form is along the lines of (16), (18), and (19).

1. First, suppose $\mathcal{R}$ is *rational of degree* N in x^2:

$$\mathcal{R}_N(x) = \sum_{m=1}^{2N} \frac{\alpha_m}{\beta_m - x}, \qquad \mathcal{R}_N^{(j)}(0) = \sum_{m=1}^{2N} \frac{j!\,\alpha_m}{(\beta_m)^{j+1}},$$

having $\mathcal{R}_N(0) = 1$, and with coefficients determined by

$$\begin{aligned}\mathcal{W}^{(2j)}(0) &= (2j-1)!\,\gamma_{2j}\,\zeta(2j+4)\\ &= \alpha_0\,\mathcal{R}_N^{(2j)}(0)\,\zeta(2j+4).\end{aligned}$$

Thus, $\alpha_0 = 17/36$ and the conditions to be met are that for some N

$$\gamma_j = \frac{17}{36} \sum_{m=1}^{2N} \frac{\alpha_m}{(\beta_m)^{j+1}}$$

for $j = 1, 2, .., N$ with $\gamma_{2j+1} \equiv 0$.

- this does not *appear* to be solvable

2. We next look for a rational *poly-exponential* generating function in which

$$\mathcal{R}_N(x) = \frac{\sum_{i=1}^N p_i(x) e^{\lambda_i x}}{\sum_{i=1}^N q_i(x) e^{\mu_i x}},$$

for polynomials p_i, q_i and scalars λ_i, μ_i, as is the case for the *Bernoulli numbers* $(t/(\exp(t)-1))$, *Euler numbers* $(2\,\mathrm{sech}(x))$ and on.

CONCLUDING COMMENTS

We believe that this general experimental procedure will ultimately yield results for yet other classes of arguments, such as for $\zeta(4n+m)$, $m=0$ or $m=1$, but our current experimental results are negative.

I. Considering $\zeta(4n+1)$, for $n=9$ the simplest evaluation we know is

$$\begin{aligned}
\zeta(9) &= \frac{9}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^9\binom{2k}{k}} - \frac{5}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2} \\
&+ 5\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4} \\
&+ \frac{45}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^6} - \frac{25}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\sum_{j=1}^{k-1}\frac{1}{j^2},
\end{aligned}$$

This is one term shorter than the 'new' identity for $\zeta(9)$ given by Rivoal, which comes from taking the coefficient of x^2y^4 in (18).

II. For $\zeta(2n+4)$ (and $\zeta(4n)$) starting with (11) which we again recall:

$$\zeta(4) = \frac{36 \cdot 1}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}},$$

the identity for $\zeta(6)$ most susceptible to bootstrapping is

$$\zeta(6) = \frac{36 \cdot 8}{163} \left[\sum_{k=1}^{\infty} \frac{1}{k^6 \binom{2k}{k}} + \frac{3}{2} \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4} \right]$$

- For $\zeta(8)$—and $\zeta(10)$—we have enticingly found:

$$\zeta(8) = \frac{36 \cdot 64}{1373} \left[\sum_{k=1}^{\infty} \frac{1}{k^8 \binom{2k}{k}} + \frac{9}{4} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4} + \frac{3}{2} \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^6} \right]$$

  – but this pattern is not fruitful; it stops after one more case ($n = 10$).

Enter RAMANUJAN Again

Hyperbolic series connect $\zeta(2N+1)$ and π^{2N+1}

- For $M \equiv -1 \pmod 4$

$$\zeta(4N+3) = -2\sum_{k\geq 1}\frac{1}{k^{4N+3}\left(e^{2\pi k}-1\right)}$$

$$+\frac{2}{\pi}\left\{\frac{4N+7}{4}\zeta(4N+4) - \sum_{k=1}^{N}\zeta(4k)\zeta(4N+4-4k)\right\}$$

where the interesting term is the hyperbolic series.

- Correspondingly, for $M \equiv 1 \pmod 4$

$$\zeta(4N+1) = -\frac{2}{N}\sum_{k\geq 1}\frac{(\pi k+N)e^{2\pi k}-N}{k^{4N+1}(e^{2\pi k}-1)^2}$$

$$+\frac{1}{2N\pi}\left\{(2N+1)\zeta(4N+2)+\sum_{k=1}^{2N}(-1)^k 2k\zeta(2k)\zeta(4N+2-2k)\right\}.$$

  - Only a finite set of $\zeta(2N)$ values is required and the full precision value e^{π} is reused throughout.

  ◇ e^{π} is the easiest transcendental to fast compute (by elliptic methods). One "differentiates" $e^{-s\pi}$ to obtain π (via the AGM iteration).

- For $\zeta(4N+1)$, I decoded "nicer" series from a couple of PSLQ observations by Simon Plouffe.

THEOREM. For $N = 1, 2, \ldots$

$$\left\{2-(-4)^{-N}\right\}\sum_{k=1}^{\infty}\frac{\coth(k\pi)}{k^{4N+1}} \;-\; (-4)^{-2N}\sum_{k=1}^{\infty}\frac{\tanh(k\pi)}{k^{4N+1}} = Q_N \times \pi^{4N+1}, \qquad (30)$$

where the quantity Q_N in (30) is an explicit rational:

$$Q_N := \sum_{k=0}^{2N+1}\frac{B_{4N+2-2k}B_{2k}}{(4N+2-2k)!(2k)!} \times \left\{(-1)^{\binom{k}{2}}(-4)^N 2^k + (-4)^k\right\}.$$

- On substituting

$$\tanh(x) = 1 - \frac{2}{\exp(2x)+1}$$

and

$$\coth(x) = 1 + \frac{2}{\exp(2x)-1}$$

one may solve for

$$\zeta(4N+1).$$

★★ We finish with two examples:

$$\zeta(5) = \frac{1}{294}\pi^5 - \frac{2}{35}\sum_{k=1}^{\infty}\frac{1}{(1+e^{2k\pi})k^5} + \frac{72}{35}\sum_{k=1}^{\infty}\frac{1}{(1-e^{2k\pi})k^5}.$$

and

$$\zeta(9) = \frac{125}{3704778}\pi^9 - \frac{2}{495}\sum_{k=1}^{\infty}\frac{1}{(1+e^{2k\pi})k^9} + \frac{992}{495}\sum_{k=1}^{\infty}\frac{1}{(1-e^{2k\pi})k^9}.$$

- Will we ever identify universal formulae like (30) automatically? My work was highly human aided.

- How do we connect these to the binomial sums?

Knots, Pens and Cameras

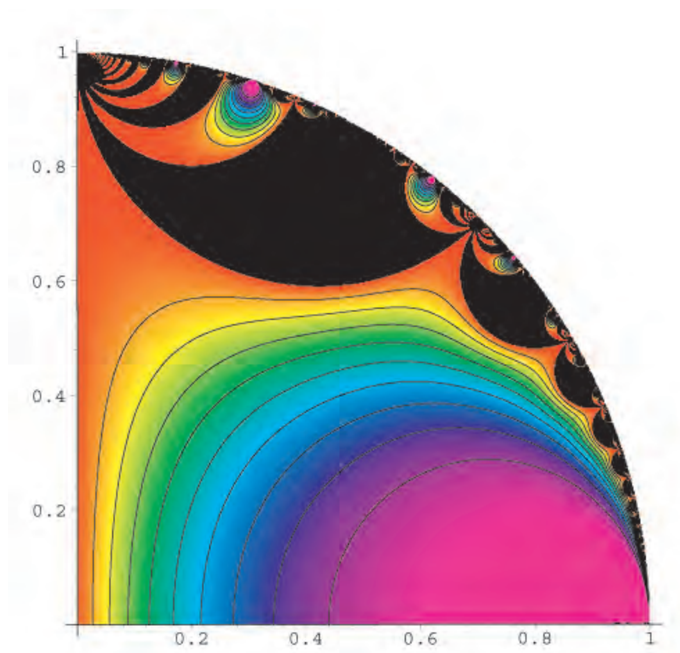
CARL FRIEDRICH GAUSS

- Boris Stoicheff's often enthralling biography of Herzberg* records Gauss writing:

It is not knowledge, but the act of learning, not possession but the act of getting there which generates the greatest satisfaction.

Carl Friedrich Gauss (1777-1855)

Fractals in **Gauss' discovery of modularity in theta functions (k=k(q))**



*Gerhard Herzberg (1903-99) fled Germany for Saskatchewan in 1935 and won the 1971 Chemistry Nobel

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