A Cyclic Douglas–Rachford Iteration Scheme in Hilbert spaces

Brailey Sims

Computer Assisted Research Mathematics & Applications
University of Newcastle
http://carma.newcastle.edu.au/

Workshop on
SPECIAL WORKSHOP ON FIXED POINT THEORY AND RELATED TOPICS
29 – 30 April MMXIII Chiang Mai University, Thailand

Abstract

What we present here is based on recent work by one of our PhD students; **Matthew K. Tam**

Abstract:

In the Hilbert space setting we present a new iteration scheme, inspired by the 2-set Douglas–Rachford scheme, but which is applicable to N-set convex feasibility problems.

Our main result is weak convergence of the method to a point whose nearest point projections onto each of the N sets coincide.

In the case of affine subspaces, norm convergence is obtained.

These results will appear in the Journal of Optimization – Theory and Applications.

Abstract

What we present here is based on recent work by one of our PhD students; **Matthew K. Tam**

Abstract:

In the Hilbert space setting we present a new iteration scheme, inspired by the 2-set Douglas–Rachford scheme, but which is applicable to N-set convex feasibility problems.

Our main result is weak convergence of the method to a point whose nearest point projections onto each of the N sets coincide.

In the case of affine subspaces, norm convergence is obtained.

These results will appear in the Journal of Optimization – Theory and Applications.

Abstract

What we present here is based on recent work by one of our PhD students; **Matthew K. Tam**

Abstract:

In the Hilbert space setting we present a new iteration scheme, inspired by the 2-set Douglas–Rachford scheme, but which is applicable to N-set convex feasibility problems.

Our main result is weak convergence of the method to a point whose nearest point projections onto each of the N sets coincide.

In the case of affine subspaces, norm convergence is obtained.

These results will appear in the Journal of Optimization – Theory and Applications.

Abstract

What we present here is based on recent work by one of our PhD students; **Matthew K. Tam**

Abstract:

In the Hilbert space setting we present a new iteration scheme, inspired by the 2-set Douglas–Rachford scheme, but which is applicable to N-set convex feasibility problems.

Our main result is weak convergence of the method to a point whose nearest point projections onto each of the N sets coincide.

In the case of affine subspaces, norm convergence is obtained.

These results will appear in the Journal of Optimization – Theory and Applications.

Abstract

What we present here is based on recent work by one of our PhD students; **Matthew K. Tam**

Abstract:

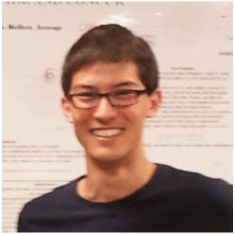
In the Hilbert space setting we present a new iteration scheme, inspired by the 2-set Douglas–Rachford scheme, but which is applicable to N-set convex feasibility problems.

Our main result is weak convergence of the method to a point whose nearest point projections onto each of the N sets coincide.

In the case of affine subspaces, norm convergence is obtained.

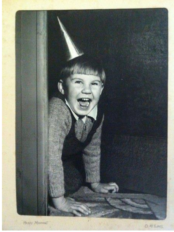
These results will appear in the Journal of Optimization – Theory and Applications.

Those involved



Matt Tam

Jon Borwein



Brailey Sims

The problem

Given N closed, convex sets with nonempty intersection, the *N-set convex feasibility problem* asks for a point contained in the intersection of the N sets.

Many optimization problems can be cast in this framework, either directly or as a suitable relaxation if a desired bound on the quality of the solution is known *a priori*.

The problem

Given N closed, convex sets with nonempty intersection, the N*-set convex feasibility problem* asks for a point contained in the intersection of the N sets.

Many optimization problems can be cast in this framework, either directly or as a suitable relaxation if a desired bound on the quality of the solution is known *a priori*.

Projection algorithms I

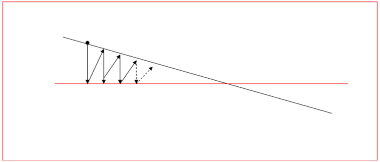
A common approach to solving N-set convex feasibility problems is the use of *projection algorithms*.

Some well known projection methods include:

Projection algorithms II

- von Neumann's alternating projections method

John von Neumann

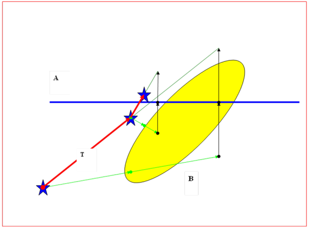


Projection algorithms

- the Douglas–Rachford method, the focus of this talk

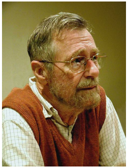
Jim Douglas

Henry H. Rachford



Projection algorithms III

- Dykstra's projection algorithm

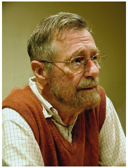


EDSGER WYBE DIJKSTRA

Of course, there are also many variants to all of these.

Projection algorithms III

- Dykstra's projection algorithm



Edsger Wybe Dijkstra

Of course, there are also many variants to all of these.

equivalences

And, on certain classes of problems, various projection methods coincide with each other, and with other known techniques. For example:

- If the sets are closed affine subspaces, alternating projections = Dykstra's method
- If the sets are hyperplanes, alternating projections = Dykstra's method = Kaczmarz's method
- If the sets are half-spaces, alternating projections = the method of Agmon, Motzkin and Schoenberg (MAMS), and Dykstra's method = Hildreth's method
- Applied to the phase retrieval problem, alternating projections = error reduction, Dykstra's method = Fienup's BIO, and Douglas–Rachford = Fienup's HIO

equivalences

And, on certain classes of problems, various projection methods coincide with each other, and with other known techniques. For example:

- If the sets are closed affine subspaces, alternating projections = Dykstra's method
- If the sets are hyperplanes, alternating projections = Dykstra's method = Kaczmarz's method
- If the sets are half-spaces, alternating projections = the method of Agmon, Motzkin and Schoenberg (MAMS), and Dykstra's method = Hildreth's method
- Applied to the phase retrieval problem, alternating projections = error reduction, Dykstra's method = Fienup's BIO, and Douglas–Rachford = Fienup's HIO

equivalences

And, on certain classes of problems, various projection methods coincide with each other, and with other known techniques. For example:

- If the sets are closed affine subspaces, alternating projections = Dykstra's method
- If the sets are hyperplanes, alternating projections = Dykstra's method = Kaczmarz's method
- If the sets are half-spaces, alternating projections = the method of Agmon, Motzkin and Schoenberg (MAMS), and Dykstra's method = Hildreth's method
- Applied to the phase retrieval problem, alternating projections = error reduction, Dykstra's method = Fienup's BIO, and Douglas–Rachford = Fienup's HIO

equivalences

And, on certain classes of problems, various projection methods coincide with each other, and with other known techniques. For example:

- If the sets are closed affine subspaces, alternating projections = Dykstra's method
- If the sets are hyperplanes, alternating projections = Dykstra's method = Kaczmarz's method
- If the sets are half-spaces, alternating projections = the method of Agmon, Motzkin and Schoenberg (MAMS), and Dykstra's method = Hildreth's method
- Applied to the phase retrieval problem, alternating projections = error reduction, Dykstra's method = Fienup's BIO, and Douglas–Rachford = Fienup's HIO

The setting and problem

Throughout, $\mathcal{H}$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$.

We use $\stackrel{w.}{\rightharpoonup}$ to denote weak convergence,

$$x_n \stackrel{w.}{\rightharpoonup} x \quad \text{iff } \langle x_n y\rangle \to \langle x, y\rangle, \quad \text{for all } y \in \mathcal{H}.$$

We are concerned with the N-set convex feasibility problem:

$$\text{Find } \; x \in \bigcap_{i=1}^{N} C_i \neq \emptyset \; \text{ where } C_i \subseteq \mathcal{H} \text{ are closed and convex.} \quad (1)$$

The setting and problem

Throughout, $\mathcal{H}$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$.

We use $\overset{w}{\rightharpoonup}$ to denote weak convergence,

$$x_n \overset{w}{\rightharpoonup} x \quad \text{iff } \langle x_n y\rangle \to \langle x, y\rangle, \quad \text{for all } y \in \mathcal{H}.$$

We are concerned with the N-set convex feasibility problem:

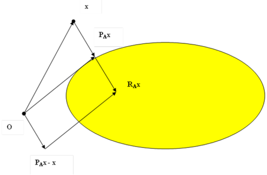
$$\text{Find} \quad x \in \bigcap_{i=1}^{N} C_i \neq \emptyset \quad \text{where } C_i \subseteq \mathcal{H} \text{ are closed and convex.} \qquad (1)$$

Projections and reflections

For a Tchebychev (hence, any nonempty closed convex) $A \subseteq \mathcal{H}$ and $x \in \mathcal{H}$ the *nearest point projection* of x onto A is,

$$P_A(x) \;:=\; \operatorname{argmin}\{\|x - c\| : c \in A\} \;=\; \{c_x\}.$$

Reflection in A is the operator $R_A : \mathcal{H} \to \mathcal{H}$ defined by $R_A := 2P_A - I$ where I denotes the *identity* operator mapping any $x \in \mathcal{H}$ to itself.



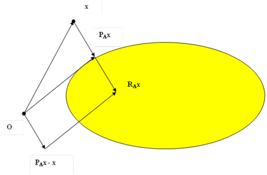
$R_A x = P_A x + (P_A x - x) = (2P_A - I)x$

Projections and reflections

For a Tchebychev (hence, any nonempty closed convex) $A \subseteq \mathcal{H}$ and $x \in \mathcal{H}$ the *nearest point projection* of x onto A is,

$$P_A(x) \;:=\; \operatorname{argmin}\{\|x - c\| : c \in A\} \;=\; \{c_x\}.$$

Reflection in A is the operator $R_A : \mathcal{H} \to \mathcal{H}$ defined by $R_A := 2P_A - I$ where I denotes the *identity* operator mapping any $x \in \mathcal{H}$ to itself.



$R_A x = P_A x + (P_A x - x) = (2P_A - I)x$

Basic facts about projections and reflections

(i) (Variational characterization of a projection)

$$P_A(x) = a_x \iff \langle x - a_x, a - a_x \rangle \leq 0 \text{ for all } a \in A.$$

(ii) (Variational characterization of a reflection)

$$R_A(x) = r \iff \langle x - r, a - r \rangle \leq \frac{1}{2}\|x - r\|^2 \text{ for all } a \in A.$$

(iii) (Translation formula) For $y \in \mathcal{H}$, $P_{y+A}(x) = y + P_A(x - y)$.

(iv) (Dilation formula) For $0 \neq \lambda \in \mathbb{R}$, $P_{\lambda A}(x) = \lambda P_A(x/\lambda)$.

(v) If A is a subspace then P_A (and hence R_A) is linear.

(vi) If A is an affine subspace then P_A (and hence R_A) is affine.

Form of projections

Application of the various iterative methods discussed above assumes that the projection onto each of the individual sets is relatively simple to compute. For instance:

Examples of projection operators P_C

- **hyperplane** $x - \frac{\langle a,x\rangle - \beta}{\|a\|^2} a$, when $C = \{x \in X \mid \langle a, x\rangle = \beta\}$;
- **positive orthant** $x_j^+ = \max\{x_j, 0\}$;
- **halfspace** $x - \frac{(\langle a,x\rangle - \beta)^+}{\|a\|^2} a$, when $C = \{x \in X \mid \langle a, x\rangle \leq \beta\}$;
- **stripes**
- **unit ball** $x/\|x\|$ if $\|x\| > 1$;
- **affine subspace** $x - A^\dagger(Ax - b)$, when $C = A^{-1}b$.
- **boxes**
- **Fourier magnitude constraints** $y(\omega)/|y(\omega)|\gamma(\omega)$
- **nearest positive semidefinite matrix** $U\Lambda^+U^T$, where $U\Lambda U^T \in \mathbb{S}^n$;
- **nearest unit vector** e_i, where $\langle e_i, x\rangle = \max_{j\in I}\langle e_j, x\rangle$;
- **dilations, translations** of the above

Nonexpansive and firmly nonexpansive maps

Let $D \subseteq \mathcal{H}$ and $T : D \to \mathcal{H}$.

We say T is *asymptotically regular* if $\|T^n x - T^{n+1} x\| \to 0$, for all $x \in \mathcal{D}$.

We denote the set of *fixed points* of T by $\operatorname{Fix} T = \{x : Tx = x\}$.
We say T is *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in D$$

We say T is *firmly nonexpansive* if

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2 \text{ for all } x, y \in D.$$

It follows that every firmly nonexpansive mapping is nonexpansive.

Basic facts for nonexpansive and firmly nonexpansive maps

(i) The class of nonexpansive maps is closed under convex combinations and compositions (this is not true for the class of firmly nonexpansive maps)

(ii) A nonexpansive self-map of a nonempty closed convex subset of $\mathcal{H}$ has a fixed point

(iii) The following are equivalent

  (a) $T : D \to \mathcal{H}$ is firmly nonexpansive
  (b) $\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2$ for all $x,\, y \in D$
  (c) $\|Tx - Ty\| \leq \|((1-t)x + tTx) - ((1-t)y + tTy)\|$ for all $t \in [0,\, 1]$ and all $x,\, y \in D$
  (d) $T = \frac{1}{2}(I + V)$, where $V : D \to \mathcal{H}$ is nonexpansive
  (e) $2T - I$ is nonexpansive

Key examples of nonexpansive and firmly nonexpansive maps

Let $A, B \subseteq \mathcal{H}$ be closed and convex. Then,

(i) P_A is firmly nonexpansive, hence

(ii) R_A is nonexpansive and

(iii) $T_{A,B} := \frac{1}{2}(I + R_B R_A)$ is firmly nonexpansive.

Some basic results I

A sufficient condition for a firmly nonexpansive map T to be asymptomatic regular is that $\mathrm{Fix}\, T \neq \emptyset$.

Although composites of firmly nonexpansive maps need not be firmly nonexpansive (even the composition of two projections onto subspaces need not be firmly nonexpansive [Censor and Reich, 1997]), this extends [Reich, 1987]:

Lemma

Let $T_i : \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive, for $i = 1, 2, \cdots, r$, and define $T := T_r \ldots T_2 T_1$. If $\mathrm{Fix}\, T \neq \emptyset$ then T is asymptotically regular.

Some basic results II

The following characterizes fixed points of certain compositions of firmly nonexpansive operators [Bauschke, 2011].

Lemma

Let $T_i : \mathcal{H} \to \mathcal{H}$ *be firmly nonexpansive, for each* i*, and define* $T := T_r \dots T_2 T_1$*. If* $F := \bigcap_{i=1}^{r} \operatorname{Fix} T_i \neq \emptyset$ *then* $\operatorname{Fix} T = F$*.*

Douglas-Rachford for two sets

The Douglas-Rachford algorithm (also known as Reflect-Reflect-Average) was introduced in 1956 in connection with numerical solutions for certain heat conduction problems. It consists of iterating the operator

$$T_{A,B} := \frac{1}{2}(I + R_B R_A) \quad (2)$$

$$= P_B(2P_A - I) + (I - P_A) \quad (3)$$

And led to:

Theorem

Let $A, B \subseteq \mathcal{H}$ be closed and convex. For any $x_0 \in \mathcal{H}$, the sequence $T^n_{A,B} x_0$ converges weakly to a point x such that $P_A x \in A \cap B$.

Douglas-Rachford for two sets

The Douglas-Rachford algorithm (also known as Reflect-Reflect-Average) was introduced in 1956 in connection with numerical solutions for certain heat conduction problems. It consists of iterating the operator

$$T_{A,B} := \frac{1}{2}(I + R_B R_A) \tag{2}$$

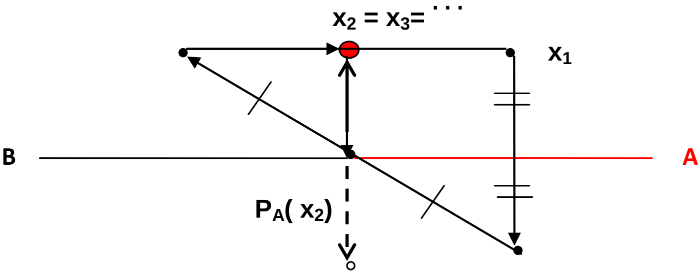
$$= P_B(2P_A - I) + (I - P_A) \tag{3}$$

And led to:

Theorem

Let $A, B \subseteq \mathcal{H}$ be closed and convex. For any $x_0 \in \mathcal{H}$, the sequence $T^n_{A,B} x_0$ converges weakly to a point x such that $P_A x \in A \cap B$.

2 set Douglas-Rachford



Lions and Mercier

This was proved by Lions and Mercier in 1979.

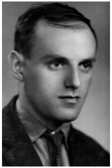
Pierre-Louis Lions **Bertrand Mercier**

Proof of D-R

There are many way to prove Theorem 3. One is to use the following well known theorem together with the facts collected above and the observation from 3 that $P_A \operatorname{Fix} T_{A,B} = A \cap B$.

Theorem (Opial, 1967)

Let $T : \mathcal{H} \to \mathcal{H}$ be nonexpansive, asymptotically regular, and $\operatorname{Fix} T \neq \emptyset$. Then for any $x_0 \in \mathcal{H}$, $T^n x_0$ converges weakly to an element of $\operatorname{Fix} T$.



Zdzislaw Opial 1930 –1974

Proof of D-R continued

Further, when T is linear, the limit can be identified and convergence is in norm.

Theorem

Let $T : \mathcal{H} \to \mathcal{H}$ be linear, nonexpansive and asymptotically regular. Then for any $x_0 \in \mathcal{H}$,

$$\lim_{n\to\infty} \|T^n x_0 - P_{\ker T} x_0\| = 0.$$

Why D-R

Interest in the Douglas–Rachford iteration is in part due to its excellent performance, despite a lack of theoretical justification, on various problems involving one or more *non-convex* sets.

For example:

- in phase retrieval problems arising in the context of image reconstruction [Bauschke, Combettes and Luke, 2002] - Hubble telescope.
- various NP-complete combinatorial problems including Boolean satisfiability and Sudoku [Elser, 2997].

In contrast, von Neumann's alternating projection method applied to such problems often fails to converge satisfactorily.

Why D-R

Interest in the Douglas–Rachford iteration is in part due to its excellent performance, despite a lack of theoretical justification, on various problems involving one or more *non-convex* sets.
For example:

- in phase retrieval problems arising in the context of image reconstruction [Bauschke, Combettes and Luke, 2002] - Hubble telescope.
- various NP-complete combinatorial problems including Boolean satisfiability and Sudoku [Elser, 2997].

In contrast, von Neumann's alternating projection method applied to such problems often fails to converge satisfactorily.

Why D-R

Interest in the Douglas–Rachford iteration is in part due to its excellent performance, despite a lack of theoretical justification, on various problems involving one or more *non-convex* sets.
For example:

- in phase retrieval problems arising in the context of image reconstruction [Bauschke, Combettes and Luke, 2002] - Hubble telescope.
- various NP-complete combinatorial problems including Boolean satisfiability and Sudoku [Elser, 2997].

In contrast, von Neumann's alternating projection method applied to such problems often fails to converge satisfactorily.

Why D-R

Interest in the Douglas–Rachford iteration is in part due to its excellent performance, despite a lack of theoretical justification, on various problems involving one or more *non-convex* sets.
For example:

- in phase retrieval problems arising in the context of image reconstruction [Bauschke, Combettes and Luke, 2002] - Hubble telescope.
- various NP-complete combinatorial problems including Boolean satisfiability and Sudoku [Elser, 2997].

In contrast, von Neumann's alternating projection method applied to such problems often fails to converge satisfactorily.

Non convex D-R

Borwein and S have provided limited theoretical justification for non-convex Douglas–Rachford iterations, proving local convergence for a prototypical instance involving a sphere and an affine subspace in Euclidean space.

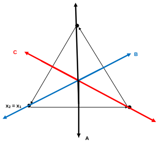
Even more recently, a local version of firm nonexpansivity has been utilized by Hesse and Luke to obtain local convergence of the Douglas–Rachford method in a limited non-convex framework.

These two sets of results are Complementary and do not directly overlap one another.

From 2 to N sets

Most projection algorithms can be extended to the N-set convex feasibility problem without significant modification.

An exception is the Douglas–Rachford method, for which only the theory of 2-set feasibility problems has so far been successfully investigated.



Divide and Concur

One approach is to reduce an $N > 2$ set problem to an equivalent 2-set feasibility problem posed in a product space to which Douglas–Rachford can be applied.

In which case the iteration effectively becomes: *'parallel' reflect in each set and then average* – a scheme also known as *divide and concur*.

How to Divide and Concur

To find a point in the intersection of N sets $A_1, A_2, \dots A_k, \dots, A_N$ in $\mathcal{H}$ we can instead consider the subset $A := \prod_{k=1}^{N} A_k$ and the (diagonal) subspace

$$B := \{x = (x_1, x_2, \dots, x_N) \colon x_1 = x_2 = \cdots = x_N\}$$

of the Hilbert space product $\prod_{k=1}^{N} \mathcal{H}$.

Then we observe that,

$$R_A : (x_1, x_2, \dots, x_N) \mapsto (R_{A_1} x_1, R_{A_2} x_2, \dots, R_{A_N} x_N),$$

so that the reflections are 'divided' up.

And,

How to Divide and Concur

To find a point in the intersection of N sets $A_1, A_2, \dots A_k, \dots, A_N$ in $\mathcal{H}$ we can instead consider the subset $A := \prod_{k=1}^{N} A_k$ and the (diagonal) subspace

$$B := \{x = (x_1, x_2, \dots, x_N) \colon x_1 = x_2 = \cdots = x_N\}$$

of the Hilbert space product $\prod_{k=1}^{N} \mathcal{H}$.

Then we observe that,

$$R_A : (x_1, x_2, \dots, x_N) \;\mapsto\; (R_{A_1}x_1, R_{A_2}x_2, \dots, R_{A_N}x_N),$$

so that the reflections are ‘divided’ up.

And,

Divide and Concur continued

$$P_B(x) = \left(\frac{x_1 + x_2 + \cdots + x_N}{N}, \ldots, \frac{x_1 + x_2 + \cdots + x_N}{N}\right),$$

so that the projection and hence reflection on B are averaging ('concurrences'); thence the name.

In this form the algorithm is particularly suited to parallelization.

N–set cyclic Douglas–Rachford

We now introduce a new projection algorithm, the *cyclic Douglas–Rachford* iteration scheme.

For $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ define $T_{[C_1\, C_2\, \ldots\, C_N]} : \mathcal{H} \to \mathcal{H}$ by

$$\begin{aligned} T_{[C_1\, C_2\, \ldots\, C_N]} &:= T_{C_N,C_1} T_{C_{N-1},C_N} \ldots T_{C_2,C_3} T_{C_1,C_2} \\ &= \left(\frac{I + R_{C_1} R_{C_N}}{2}\right) \left(\frac{I + R_{C_N} R_{C_{N-1}}}{2}\right) \ldots \left(\frac{I + R_{C_3} R_{C_2}}{2}\right) \end{aligned}$$

Given $x_0 \in \mathcal{H}$, the *cyclic Douglas–Rachford* method iterates by setting $x_{n+1} = T_{[C_1\, C_2\, \ldots\, C_N]} x_n$.

2–set cyclic Douglas–Rachford

For two sets this reduces to,

$$T_{[C_1\,C_2]} = T_{C_2,C_1} T_{C_1,C_2} = \left(\frac{I + R_{C_1} R_{C_2}}{2}\right)\left(\frac{I + R_{C_2} R_{C_1}}{2}\right).$$

So, the 2–set cyclic Douglas–Rachford scheme does not coincide with the classic Douglas–Rachford scheme.

Notation and conventions

When there is no ambiguity we abbreviate T_{C_i,C_j} by $T_{i,j}$, and $T_{[C_1\,C_2\,\ldots\,C_N]}$ by $T_{[1\,2\,\ldots\,N]}$.
Indices will always be understood modulo N. In particular, $T_{0,1} := T_{N,1}$, $T_{N,N+1} := T_{N,1}$, $C_0 := C_N$ and $C_{N+1} := C_1$.

We are now ready to present our main result, regarding convergence of the cyclic Douglas–Rachford scheme.

Notation and conventions

When there is no ambiguity we abbreviate T_{C_i,C_j} by $T_{i,j}$, and $T_{[C_1\,C_2\,\dots\,C_N]}$ by $T_{[1\,2\,\dots\,N]}$.
Indices will always be understood modulo N. In particular, $T_{0,1} := T_{N,1}$, $T_{N,N+1} := T_{N,1}$, $C_0 := C_N$ and $C_{N+1} := C_1$.

We are now ready to present our main result, regarding convergence of the cyclic Douglas–Rachford scheme.

Convergence of cyclic Douglas–Rachford

Theorem (Cyclic Douglas–Rachford)

Let $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ be closed and convex with nonempty intersection. For any $x_0 \in \mathcal{H}$, the sequence $T^n_{[1\,2\,\ldots\,N]}x_0$ converges weakly to a point x such that $P_{C_i}x = P_{C_j}x$ for all i, j. Moreover, $P_{C_j}x \in \bigcap_{i=1}^N C_i$, for each j.

Proof of cyclic Douglas–Rachford

Proof:

$T_{i,i+1}$ is firmly nonexpansive, for each i and, since $\mathrm{Fix}\, T_{i,i+1} \supseteq C_i \cap C_{i+1}$, we have $\bigcap_{i=1}^N \mathrm{Fix}\, T_{i,i+1} \supseteq \bigcap_{i=1}^N C_i \neq \emptyset$. So, $T_{[1\,2\,\dots\,N]}^n x_0$ converges weakly to a point $x \in \mathrm{Fix}\, T_{[1\,2\,\dots\,N]} = \bigcap_{i=1}^N \mathrm{Fix}\, T_{i,i+1}$.

Further, for each i, $P_{C_i} x = P_{C_i} T_{i,i+1} x \in C_i \cap C_{i+1} \subseteq C_{i+1}$.

Now compute,

Proof of cyclic Douglas–Rachford continued

$$\frac{1}{2}\sum_{i=1}^{N}\|P_{C_i}x - P_{C_{i-1}}x\|^2$$

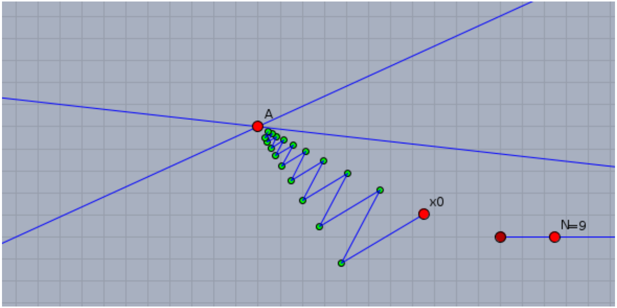
$$= \langle x, 0\rangle + \frac{1}{2}\sum_{i=1}^{N}\left(\|P_{C_i}x\|^2 - 2\langle P_{C_i}x, P_{C_{i-1}}x\rangle + \|P_{C_{i-1}}x\|^2\right)$$

$$= \left\langle x, \sum_{i=1}^{N}(P_{C_{i-1}}x - P_{C_i}x)\right\rangle - \sum_{i=1}^{N}\langle P_{C_i}x, P_{C_{i-1}}x\rangle + \sum_{i=1}^{N}\|P_{C_i}x\|^2$$

$$= \sum_{i=1}^{N}\langle x - P_{C_i}x, P_{C_{i-1}}x - P_{C_i}x\rangle \leq 0.$$

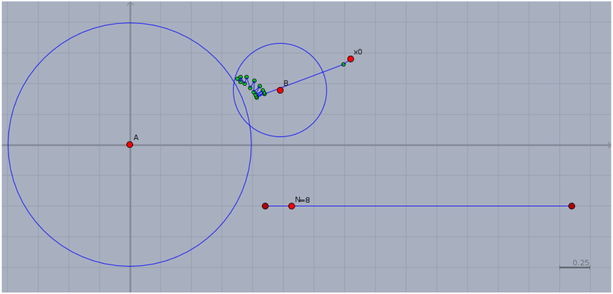
Thus, $P_{C_i}x = P_{C_{i-1}}x$, for each i. □

Illustration I



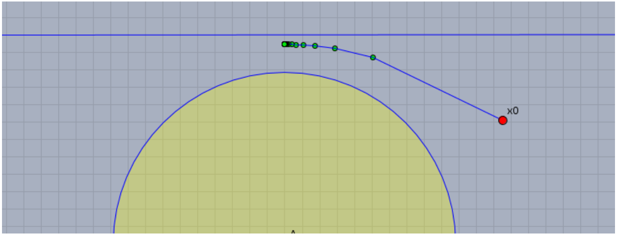
An interactive *Cinderella* applet showing a cyclic Douglas–Rachford trajectory differing from von Neumann's alternating projection method. Each green dot represents a 2-set cyclic Douglas–Rachford iteration.

Illustration II



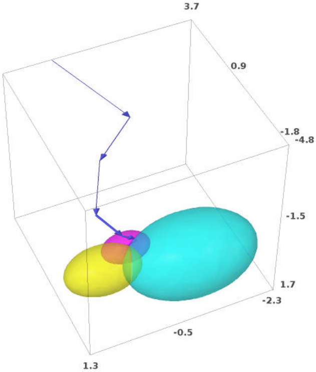
An interactive *Cinderella* applet using the cyclic Douglas–Rachford method to solve a feasibility problem with two sphere constraints. Each green dot represents a 2-set cyclic Douglas–Rachford iteration.

Illustration III



An interactive *Cinderella* applet showing the cyclic Douglas–Rachford method applied to the case of a non-intersecting ball and a line. The method appears convergent to a point whose projections onto the constraint sets form a best approximation pair. Each green dot represents a cyclic Douglas–Rachford iteration.

Illustration IV



Cyclic Douglas–Rachford algorithm applied to a 3-set feasibility problem in $\mathbb{R}^3$.

The constraint sets are colored in blue, red and yellow. Each arrow represents a 3-set cyclic Douglas–Rachford iteration.

Conclusion

Numerical experiments on instances involoving ball/sphere constraints suggest that that the cyclic Douglas–Rachford scheme outperforms divide and concur , which suffers as a result of the product formulation, although having the advantage of possible parallel implementation.

For inconsistent 2-set problems, there is evidence suggesting that the cyclic Douglas–Rachford scheme yields best approximation pairs.